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PARAMETER PLANE STUDY OF THE OPTIMAL REGULATOR

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THESIS

PARAMETER PLANE STUDY OF
THE OPTIMAL REGULATOR

by

Lionel Alfonso Doren Swett

Thesis Advisor: G.J. Thaler

JUN 1971

Approved for public release; distribution unlimited.

Parameter Plane Study of the Optimal Regulator

by

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June 1971

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ABSTRACT

Parameter plane studies of an optimal second order regulator are presented. Emphasis is placed on the interpretation of the cost function and the sensitivity of the cost function to plant parameter incremental variations. An analysis is made of cost function weighting factors and their effect on damping, speed of response, and cost.

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I. INTRODUCTION

The parameter plane study of an optimal regulator was undertaken with the purpose of finding an interpretation of the meaning of the cost function and of the weighting factors as they relate to the observed physical behaviour of the regulator.

Chapter one gives a brief description of the problem formulation; Chapter two describes some studies on the performance index; Chapter three deals with the study of the weighting factors, and Chapter four with some sensitivity aspects. Appendices A and B deal with the development of the general expression for the R matrix and the performance index.

A. PROBLEM FORMULATION

Suppose that initially the plant input or any of its derivatives is nonzero. Provide a plant input to bring the output or its derivatives to zero. In other words, the problem is to apply a control to take the plant from a nonzero state to the zero state. This problem may occur where the plant is subjected to unwanted disturbances that perturb its output (e.g., a radar antenna control system with the antenna subject to wind gusts).

The desirable properties of a regulator system should be:

A. The regulator system should involve a linear control law.

B. By definition an optimal system is one that minimizes a certain cost.

To achieve property B, let us define a performance index

$$PI = \int_{0}^{\infty} (X^{T}QX + u^{T}Pu)\ dt$$

where Q is symmetric positive definite and R is a positive definite matrix. $\int_{0}^{\infty} X^{T}QX\ dt$ come from the minimum integral-square error problem.

$$\int_{0}^{\infty} \sum_{i=1}^{n} (X_{i})^{2}\ dt = \int_{0}^{\infty} (X^{T}X)\ dt$$

constituting a reasonable measure of the system transient response, and $\int_{0}^{\infty} u^{T}Pu\ dt$ comes from the minimum energy problem.

The minimization problem, i.e., the task of finding an optimal control that minimizes the performance index, turns out to be achievable with a linear feedback law. This is the reason why the performance index includes a measure control energy.

For the infinite-time regulator problem R(t) is the solution of the reduced Riccati equation

$$R_{o}A + A^{T}R_{o} - R_{o}BP^{-1}B^{T}R_{o} + Q = 0$$

with an optimal control defined by

$$u^{*}(t) = -P^{-1}(t)B^{T}(t)R_{o}\ x(t)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -p \end{bmatrix}, \quad A^T = \begin{bmatrix} 0 & 0 \\ 1 & -p \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ K \end{bmatrix}, \quad B^T = \begin{bmatrix} 0 & K \end{bmatrix}$$

$$R_o = \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix}$$

From Appendix A the general expression for the R matrix is:

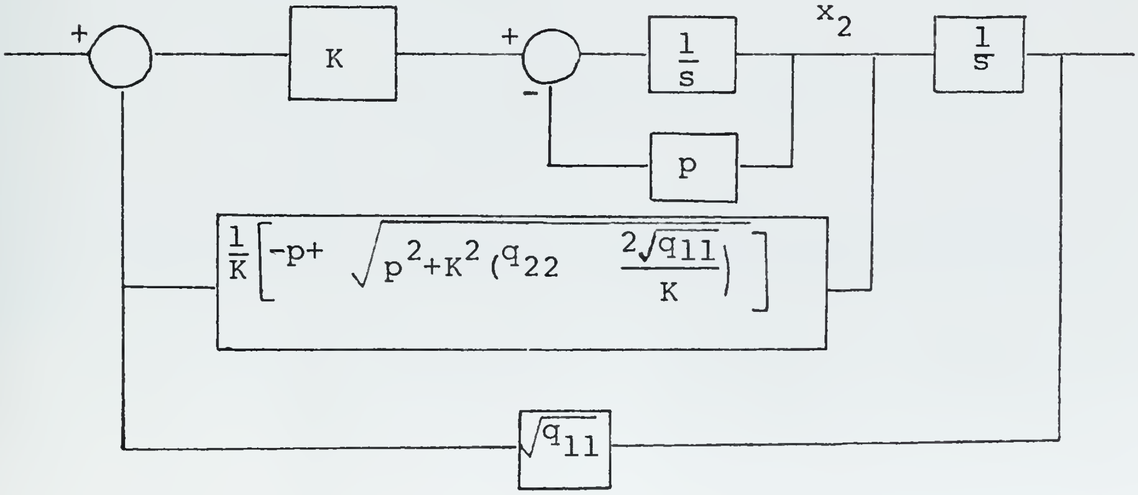
$$R_o = \begin{bmatrix} \frac{\pm p\sqrt{q_{11}}}{K} + \left(\pm\frac{\sqrt{q_{11}}}{K}\right)\left(-p + \sqrt{p^2+K^2q_{22} \pm 2Kq_{11}}\right) & \frac{\pm\sqrt{q_{11}}}{K} \\ \frac{\pm\sqrt{q_{11}}}{K} & \frac{-p}{K^2} + \sqrt{\frac{p^2}{K^4} + \frac{q_{22}}{K^2} \pm \frac{2\sqrt{q_{11}}}{K^3}} \end{bmatrix} \quad (1.1)$$

$$u_o(x) = -p^{-1}B^TR_ox$$

if $\quad P^{-1} = 1, \quad B^T = [0 \quad K]$

$$u_o(x) = [\pm\sqrt{q_{11}}\, x_1 + \left(\frac{-P}{K} + \sqrt{\frac{P^2}{K^2} + q_{22} \pm \frac{2\sqrt{q_{11}}}{K}}\right)x_2$$

from which the system block diagram is:



+
K
+
-
$\frac{1}{s}$
x_2
$\frac{1}{s}$
p
$\frac{1}{K}\left[-p+ \sqrt{p^2+K^2(q_{22} \quad \frac{2\sqrt{q_{11}}}{K})}\right]$
$\sqrt{q_{11}}$

Reducing to single block:

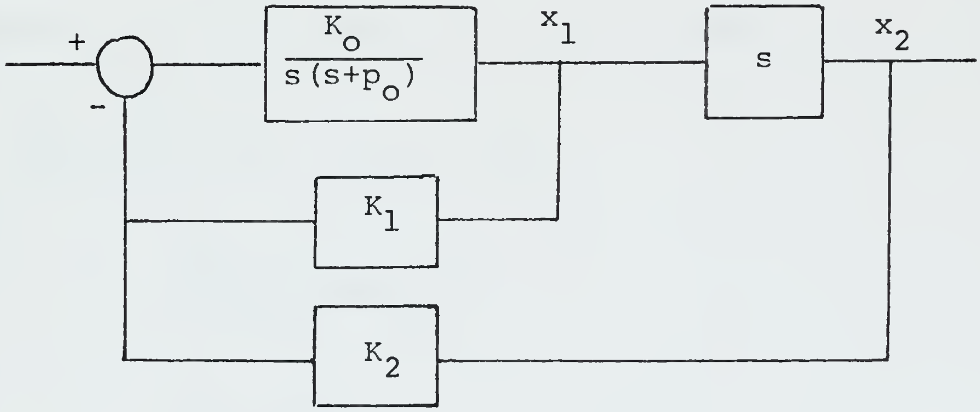
$$\boxed{\frac{K}{s^2 + s\sqrt{p^2 + K^2(q_{22} \pm \frac{2\sqrt{q_{11}}}{K}) + K\sqrt{q_{11}}}}}$$

$$\omega_n = \sqrt{K\sqrt{q_{11}}} \tag{1.2}$$

$$\zeta = \frac{1}{2\sqrt{K\sqrt{q_{11}}}} \sqrt{p^2 + K^2(q_{22} \pm \frac{2\sqrt{q_{11}}}{K})}$$

$$= \frac{1}{2}\sqrt{\frac{p^2}{K\sqrt{q_{11}}} + \frac{K^2 q_{22}}{K\sqrt{q_{11}}} \pm 2} \tag{1.3}$$

B. OPTIMAL SECOND ORDER SYSTEM



K and p are nominal values and K_1 and K_2 are adjusted so that

$$K_1 = \sqrt{\frac{q_{11}}{p_{11}}}, \quad K_2 = \frac{1}{K_o}\left[-p_o + \sqrt{p_0^2 + K_0^2\frac{q_{22}}{p_{11}} \pm 2K_o\sqrt{\frac{q_{11}}{p_{11}}}}\right] \quad (1.4)$$

r_{11} and r_{12} defined as in 1.1.

The cost function for the optimal regulator is given by:

$$J_o = r_{11}x_1^2(0) + 2r_{12}x_1(o)x_2(o) + r_{22}\, x_2^2(o)$$

$$\therefore \frac{J_o}{p_{11}} = \frac{1}{K_o}\sqrt{\frac{q_{11}}{p_{11}}}\left[\sqrt{p_o^2 + K_o^2\left(\frac{q_{22}}{p_{11}}\right) \pm 2K_o\sqrt{\frac{q_{11}}{p_{11}}}}\right]x_1^2(0)$$

$$+ \frac{2}{K_0}\sqrt{\frac{q_{11}}{p_{11}}}\; x_1(o)x_2(0)$$

$$+ \frac{1}{K_o^2}\left[-p_o + \sqrt{p_o^2 + K_o^2\left(\frac{q_{22}}{q_{11}}\right) + 2K_o\sqrt{\frac{q_{11}}{p_{11}}}}\right]x_2^2(0)$$

And substituting the values of K_1 and K_2 gives

$$\frac{J_o}{p_{11}} = \frac{K_1}{K_o}\,(p_o + K_2K_o)\; x_1^2(0)$$

$$+ \frac{2K_1}{K_o}\, x_1(0)x_2(0)$$

$$+ \frac{K_2}{K_o}\, x_2^2(0)$$

II. THE PERFORMANCE INDEX

A. GENERAL

In Appendix B the general expression of the performance index for a second order system has been derived, giving, for initial conditions on X only

$$\frac{J_o}{[x_1(0)]^2} = \left\{ (K_1^2 + \frac{q_{11}}{p_{11}}) + \frac{(K_1^2 + \frac{q_{11}}{p_{11}})(p_o + K_2K_o)^2}{K_1K_0} + K_1K_o(K_2^2 + \frac{q_{22}}{p_{11}}) - 2K_1K_2(p_o + K_2K_o) \right\} \frac{1}{2(p_o + K_2K_o)} \qquad (2.1)$$

where K_o is the plant pole,

K_1 and K_2 are the gains of the feedback paths,

q_{11}, q_{22}, p_{11} are weighting factors.

Let $q_{11} = 1$, $q_{22} = 0$, $p_{11} = 1$, $K_o = 1600$ and $p_o = 30$

then $\frac{J}{[x_1(0)]^2} = 0.04002$.

When $q_{22} = 0$, then equation (1.4) can be written as:

$$K_o = \frac{-2}{K_2} p_o + \frac{2}{K_2^2}$$

which is a straight line on the K vs. p plane. Figure 2.1 gives the values of K required to optimize the regulator. For example K = .02 is necessary to optimize a regulator with a plant pole of 30 and $\omega_n^2 = 2000$.

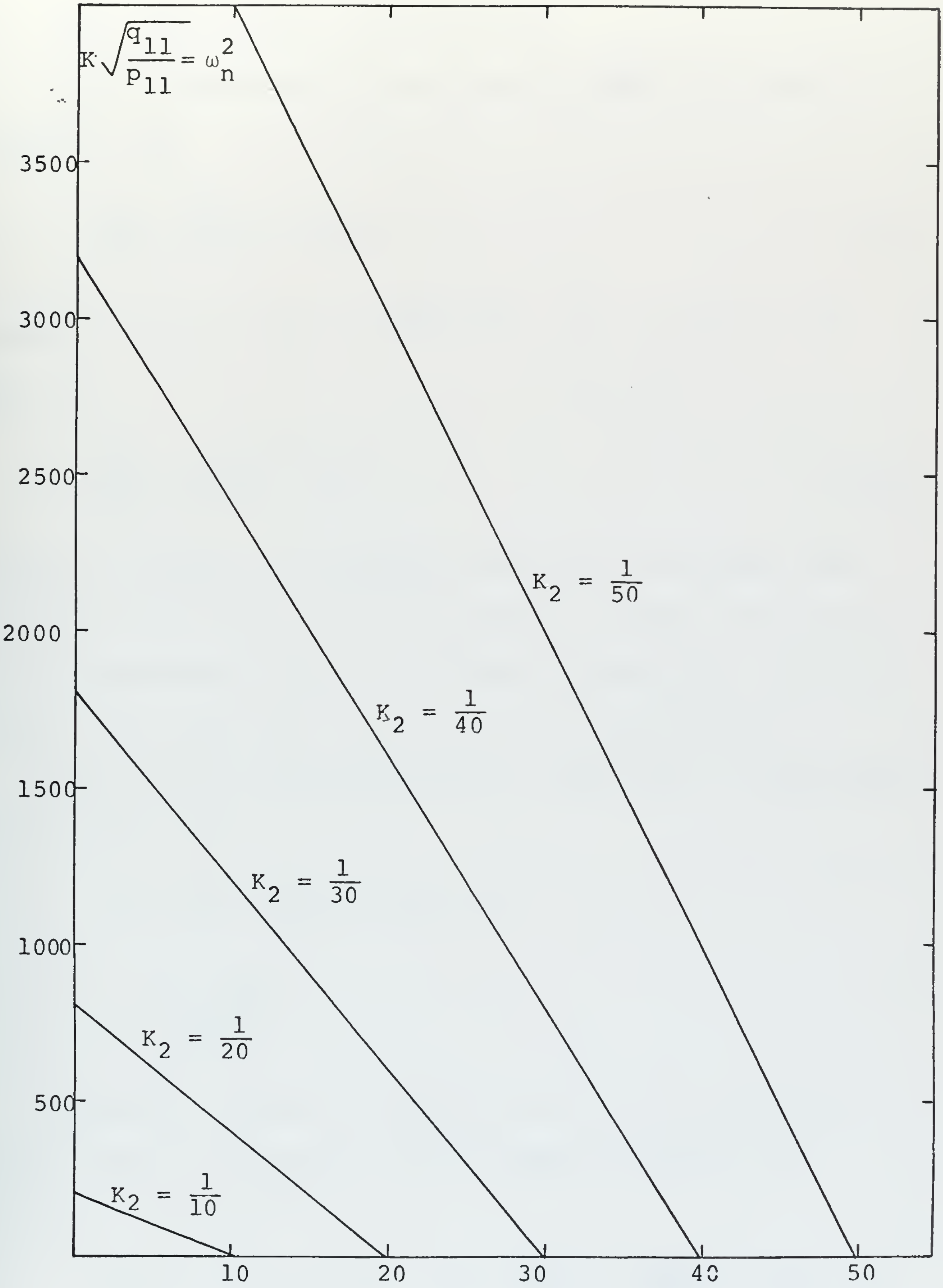


Figure 2.1. Required values of K_2 to optimize the regulator.

B. STUDY OF J ON THE PARAMETER PLANE

The cost function for the optimal system was given in section A by

$$\frac{J_o}{p_{11}} = \frac{K_1}{K}(p + K_2K) \qquad (2.2)$$

substituting $K_2 = \frac{1}{K}[-p + \sqrt{p^2 + 2k}\ ]$ into 2.2 gives, for $q_{22} = 0$

$$\frac{J_o}{p_{11}} = \frac{1}{K}\sqrt{p^2 + 2K} \qquad (2.3)$$

Figure 2.2 shows the parameter plane for the optimal cost function obtained by varying K_2 when p changes, with K as a family parameter and $K_1 = 1$. Lines of constant p and constant K have been drawn for clarity of presentation.

The value of ζ corresponding to each point in the plane is obtained from

$$s^2 + (p + KK_2)s + KK_1 = 0$$

$$\zeta = \frac{p + KK_2}{2\sqrt{KK_1}}$$

A plot of constant ζ curves on the K_1, K_2 parameter plane for K = 1600 and p = 30 is shown in Figure 2.3. From 2.1, for $q_{22} = 0$

$$KK_2^2 + (2p - 2KJ_1)K_2 + (K_1^2 + \frac{q_{11}}{p_{11}})^2 + \frac{2p^2}{1600} - 2pJ_1 = 0.$$

Choosing p = 30, K = 1600, $q_{11} = p_{11} = 1$, the contour in J_1 - K_2 plane is described by

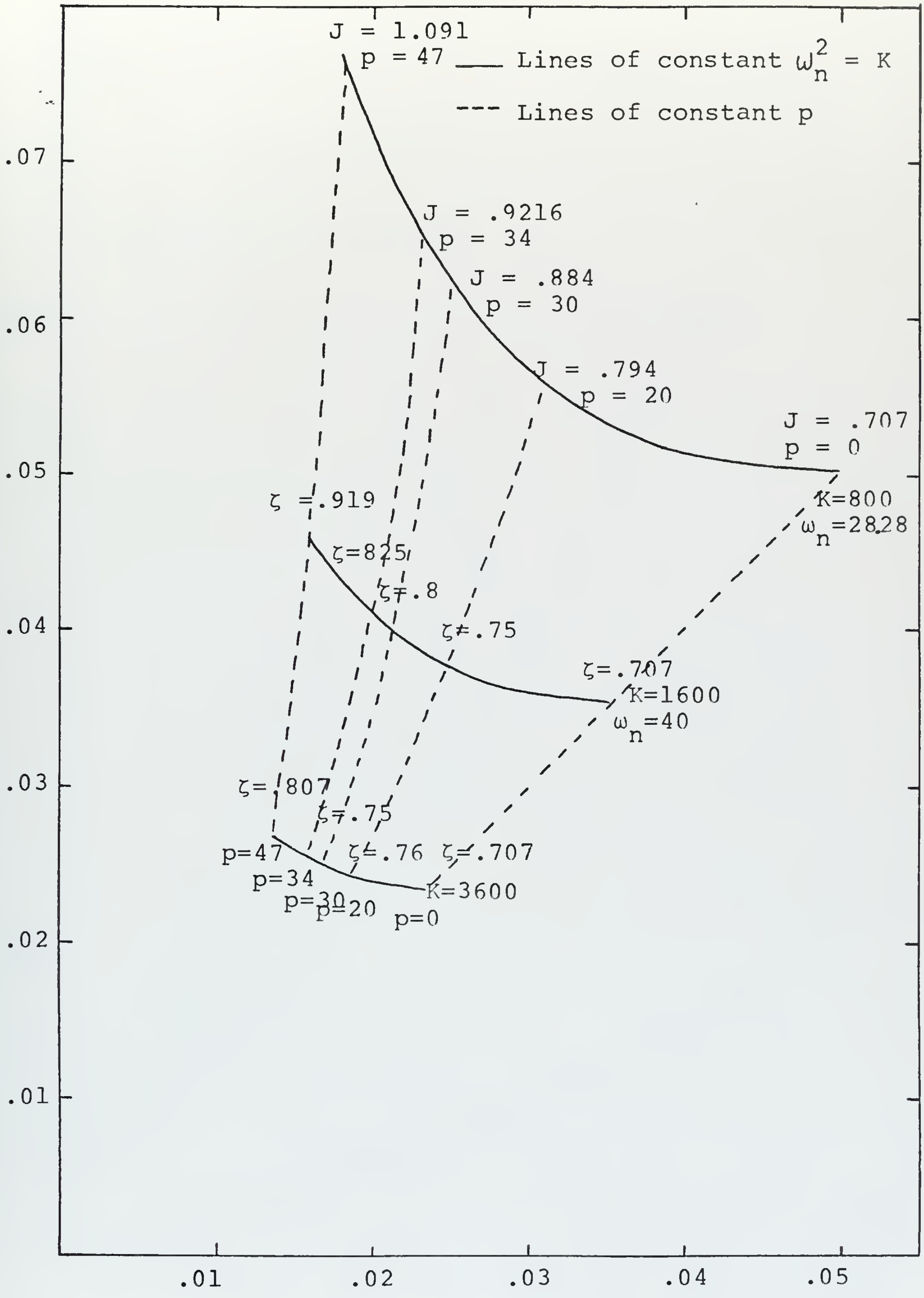


Figure 2.2. Loci of optimal cost function, K_1 = 1.0.

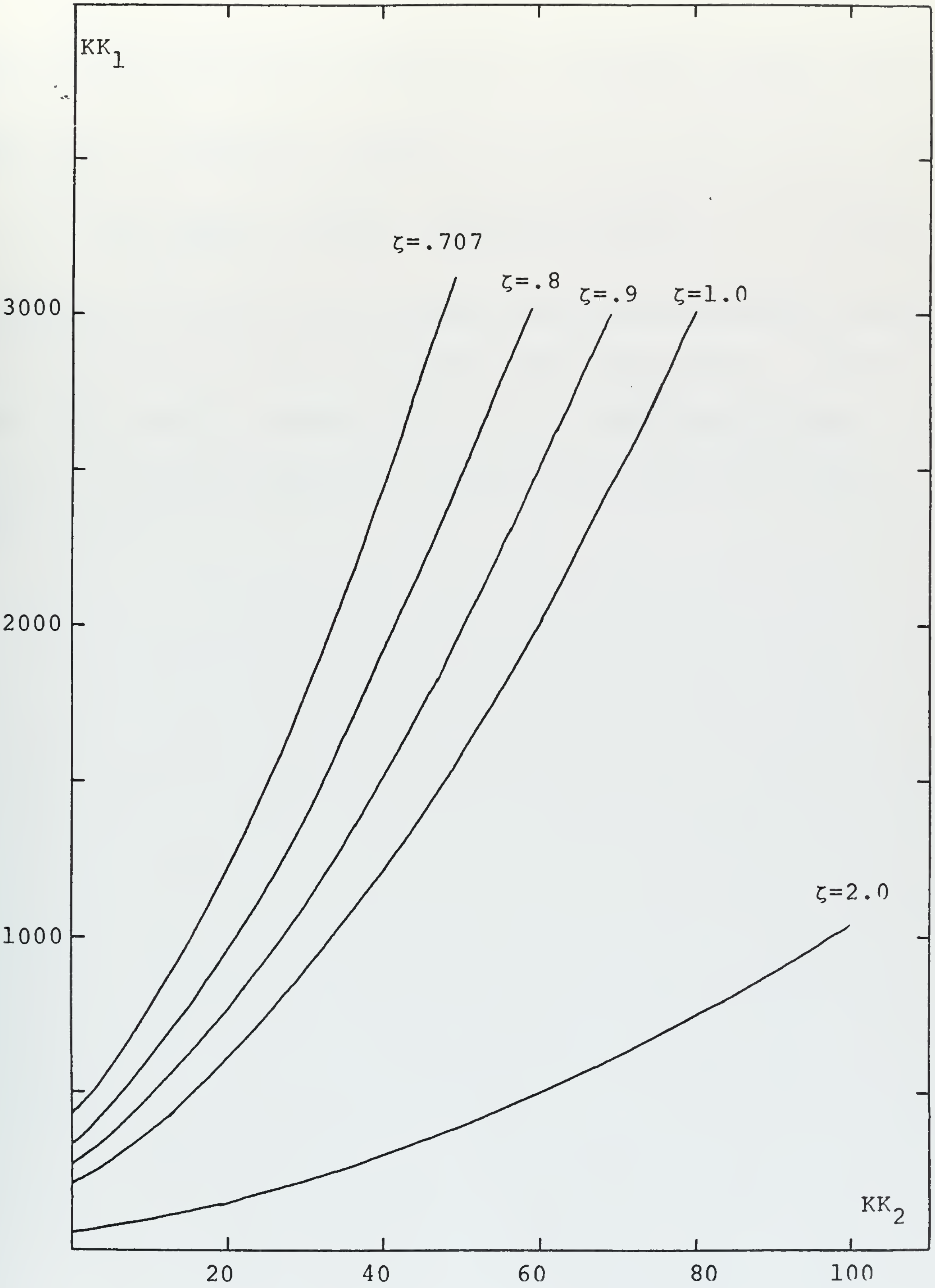


Figure 2.3. Constant ζ curves when $K = 1600$, $p = 30$.

$$K_2 = \frac{-(.0375 - 2J_1)}{2} \pm \frac{1}{2}\sqrt{(.0375-2J_1)^2 - 4(.001953-.0375J_1)} \qquad (2.4)$$

and if K_2 = constant = .02125

$$J_1 = \frac{1}{128}\left[1 + \frac{2.56}{K_1} + .56256\ K_1 + K_1^2\right] \qquad (2.5)$$

Equations 2.4 and 2.5 are represented on the parameter plane in Figure 2.4. It can be readily seen that for $K_2 = .02125$ the curve shows a minimum cost of .04. The same minimum cost is obtained keeping K_2 constant at a value of .02125 for $K_1 = 1$.

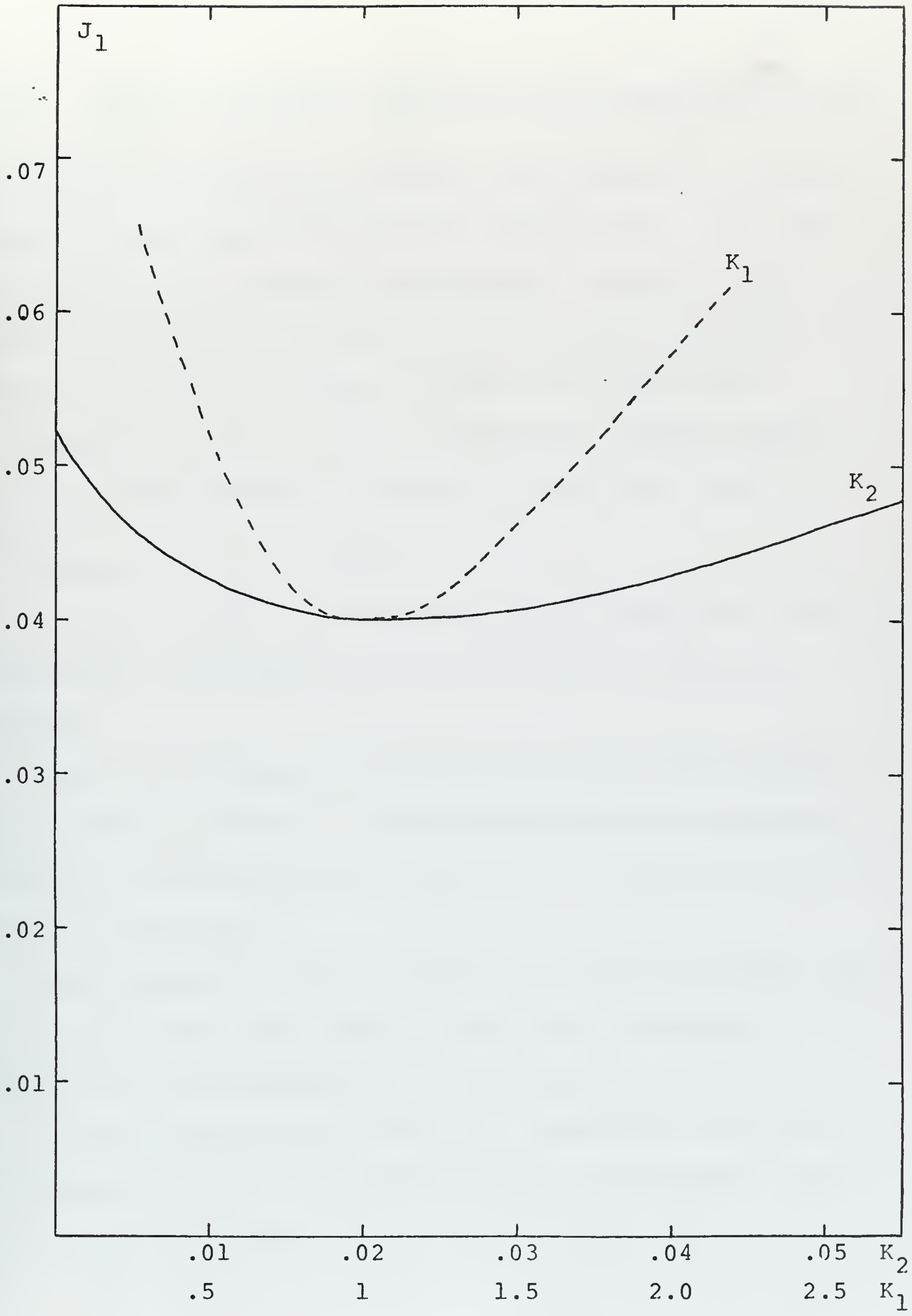


Figure 2.4. Optimal cost function given K_1 or K_2.

III. STUDY OF WEIGHTING FACTORS OF THE PERFORMANCE INDEX

When it is desired to optimize the design of a control system, a cost functional must be established. This cost functional is an analytical expression, related to the system, which must be minimized by choosing parameters which will cause an extremal of the cost functional.

Weighting factors used in performance indices determine to a large extent the optimal system which results.

A. SELECTION OF THE Q MATRIX

Optimal control law determination by Athans and Falb [1] require that the weighting factors matrix be positive definite.

Tuel [2] developed a canonical form for the weighting factor matrix Q which has the minimum number of parameters required in the performance index for the computation of the optimal control law.

The selection of this Q matrix is a very important part of optimal design but there is very little guidance in the literature in the selection of the weighting factors.

In any regulator the output or "controlled variable" is of primary interest and certainly must be weighted, being never negative. Then

$$q_{11} > 0$$

The velocity ($x_2 = \dot{x}_1$) need not be weighted at all but certainly any weight attributed to the velocity will

alter the dynamic response and may give desirable features. Thus

$$q_{22} > 0$$

But we should normally expect $q_{22} < q_{11}$.

B. DERIVATION OF EXPRESSION FOR STUDY IN THE PARAMETER PLANE

For the second order system the characteristic equation is:

$$s^2 + (p_o + K_2 K_o)s + K_o K_1 = 0$$

where

$$K_1 \triangleq \pm\sqrt{\frac{q_{11}}{p_{11}}}$$

$$K_2 = -\frac{p_o}{K_o} + \frac{1}{K_o}\sqrt{p_o^2 + K_o^2 \frac{q_{22}}{p_{11}} \pm 2K_o\sqrt{\frac{q_{11}}{p_{11}}}}$$

then

$$p_o + K_o K_2 = \sqrt{p_o^2 + K_o^2 \frac{q_{22}}{p_{11}} \pm 2K_o\sqrt{\frac{q_{11}}{p_{11}}}} = 2\zeta\omega_n$$

$$K_o K_1 = \omega_n^2 = K_o\sqrt{\frac{q_{11}}{p_{11}}}$$

Let

$$K_o\sqrt{\frac{q_{11}}{p_{11}}} = y \ , \quad \frac{q_{22}}{p_{11}} = x$$

then

$$4\zeta^2\omega_n^2 = p_o^2 + K_o^2 x + 2y.$$

It can be noted that the constant ζ curves are straight lines, but the location of the family for $0 < \zeta < 1$ depends on K_o and p_o, i.e., the effect of q_{11} and q_{22} can be shown only for a specific plant.

Figure 3.1 shows the parameter plane for the plant with $K_o = 1600$, $p_o = 30$.

From section B the cost function for the optimal system was:

$$\frac{J_o}{p_{11}} = \frac{1}{K_o} \quad \frac{q_{11}}{p_{11}} \left[\sqrt{p_o^2 + K_o^2 \left(\frac{q_{22}}{p_{11}}\right) \pm 2K_o \sqrt{\frac{q_{11}}{p_{11}}}} \right] x_1^2(0)$$

$$+ \frac{2}{K_o} \sqrt{\frac{q_{11}}{p_{11}}} \quad x_1(0) x_2(0)$$

$$+ \frac{1}{K_o^2} \left[-p_o + \sqrt{p_o^2 + K_o \left(\frac{q_{22}}{p_{11}}\right) \pm 2K_0 \sqrt{\frac{q_{11}}{p_{11}}}} \right] x_2^2(0)$$

Let

$$J_1 = \frac{J_o}{p_{11}[x_1^2(0)]}$$

$$K_0^2 J_1 = K_o \quad \frac{q_{11}}{p_{11}} \sqrt{p_o^2 + K_o^2 \left(\frac{q_{22}}{p_{11}}\right) \pm 2K_o \quad \frac{q_{11}}{p_{11}}}$$

$$= y \sqrt{p_o^2 + K_o^2 x \pm 2y}$$

For the optimal system:

$$J_1 = .04, \quad K_o = 1600, \quad p_o = 30.$$

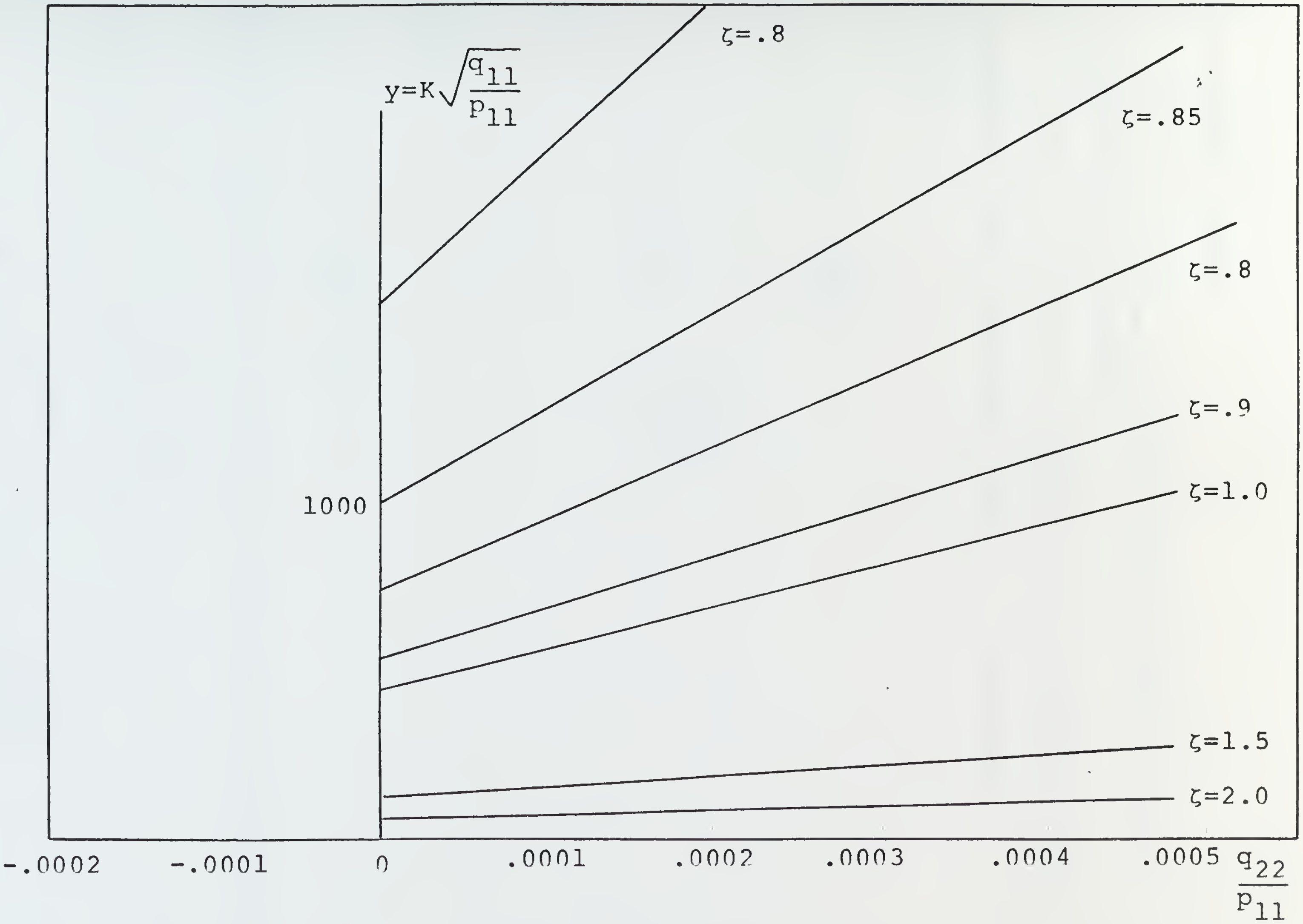


Figure 3.1. Constant curves on q_{11} - q_{22} parameter plane.

A constant cost curve is shown in Figure 3.2 in the $\frac{q_{22}}{p_{11}}$ vs. ω_n^2 parameter plane.

C. EFFECT ON DAMPING AND COST, STEADY STATE ACCURACY AND SPEED OF RESPONSE

In order to study the effect of the weighting factors on damping in the parameter plane it is necessary to use 1.2 and 1.3.

$$\omega_n^2 = K\sqrt{\frac{q_{11}}{p_{11}}} = z$$

$$\zeta = \frac{1}{2}\sqrt{\frac{p^2}{z} + \frac{K^2\frac{q_{22}}{p_{11}}}{z} \pm 2}$$

let

$$q_{22} = Wq_{11}$$

then

$$\zeta = \frac{1}{2}\sqrt{\frac{p^2}{z} + Wz \pm 2} \qquad (3.1)$$

From this equation the following expression can be obtained:

$$p^2 = -z(\pm 2 - 4\zeta^2) - Wz^2 \qquad (3.2)$$

As K is defined in Section A as:

$$K_2 = -\frac{p}{K} + \frac{1}{K}\sqrt{p^2 + K^2\frac{q_{22}}{p_{11}} \pm 2K\sqrt{\frac{q_{11}}{p_{11}}}}$$

where

$$K_1 = \pm\sqrt{\frac{q_{11}}{p_{11}}}$$

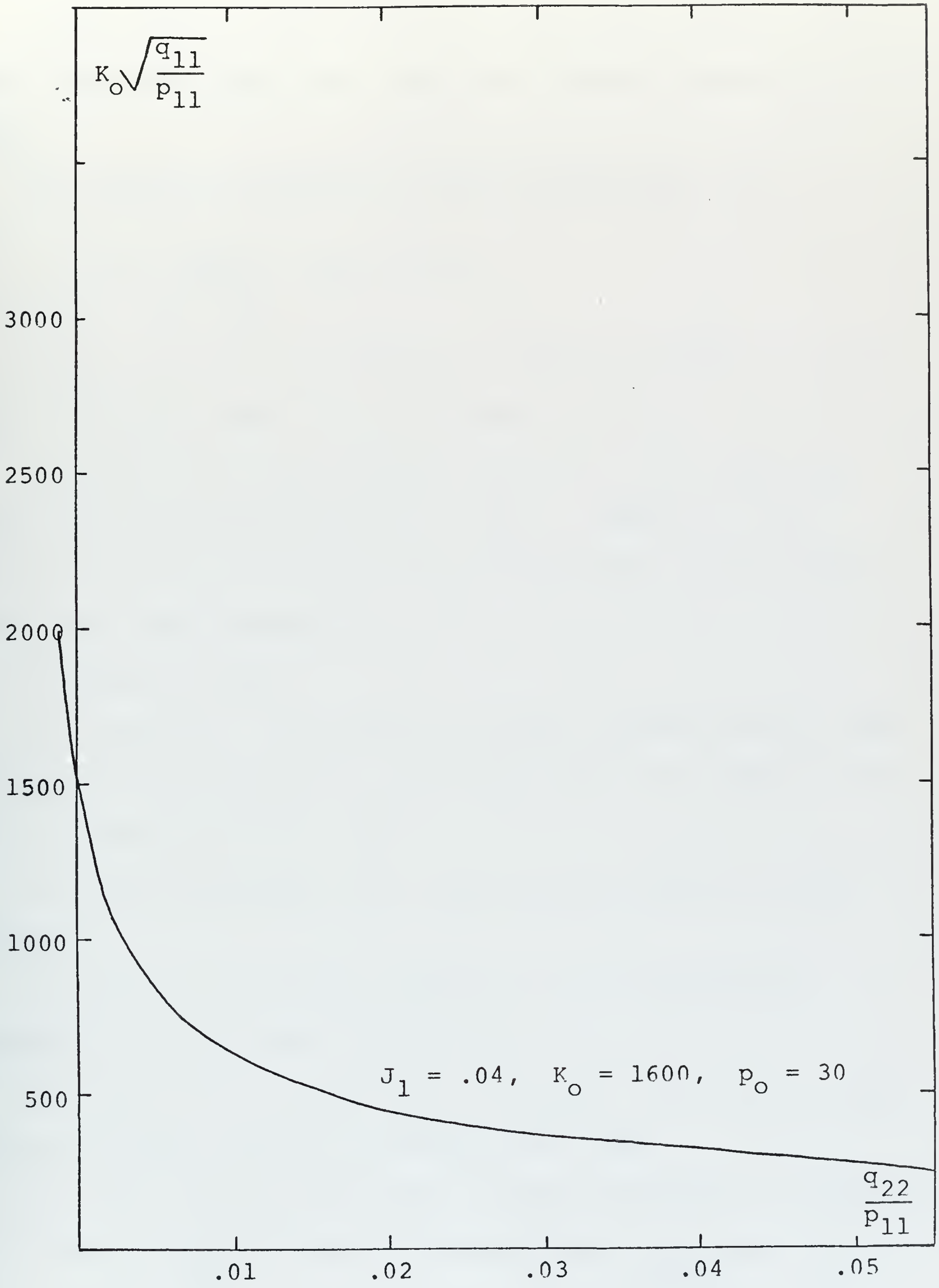


Figure 3.2. Cost function curve when $K_o = 1600$, $p_o = 30$.

Then, for $q = 0$ the choice of + sign will require negative feedback, while the - sign will require positive feedback.

For negative feedback 3.2 can be rearranged to:

$$x = \frac{-p^2}{2 - 4\zeta} \quad \text{for} \quad W = 0.$$

Then, as z must be positive, ζ must be greater or equal to .707 noting that if $\zeta = .707$ then $p = 0$.

In Figure 3.3 the parameter plane for $W = 0$ has been drawn for different values of ζ and in Figure 3.4 for $W \neq 0$. In both figures, curves of constant cost function have been superimposed.

It can be noted:

1. Minimum ζ for optimal regulator (second order) with negative feedback is .707, and this occurs only when $W = 0$.

2. For $W > 0$, $\zeta > .707$.

3. As W increases, the system rapidly becomes overdamped.

4. If p is large the system will be very heavily damped, no matter what the gain is.

5. For $W > 0$ if the gain is raised, the system ζ goes thru a minimum and then increases with gain.

6. The larger W, the more velocity feedback is required.

For plotting of cost functions, K has been kept constant at 1600 and $\sqrt{\frac{q_{11}}{p_{11}}}$ varied in Figure 3.3a and $\sqrt{\frac{q_{11}}{p_{11}}}$ kept constant at 1 and K varied in Figure 3.3b.

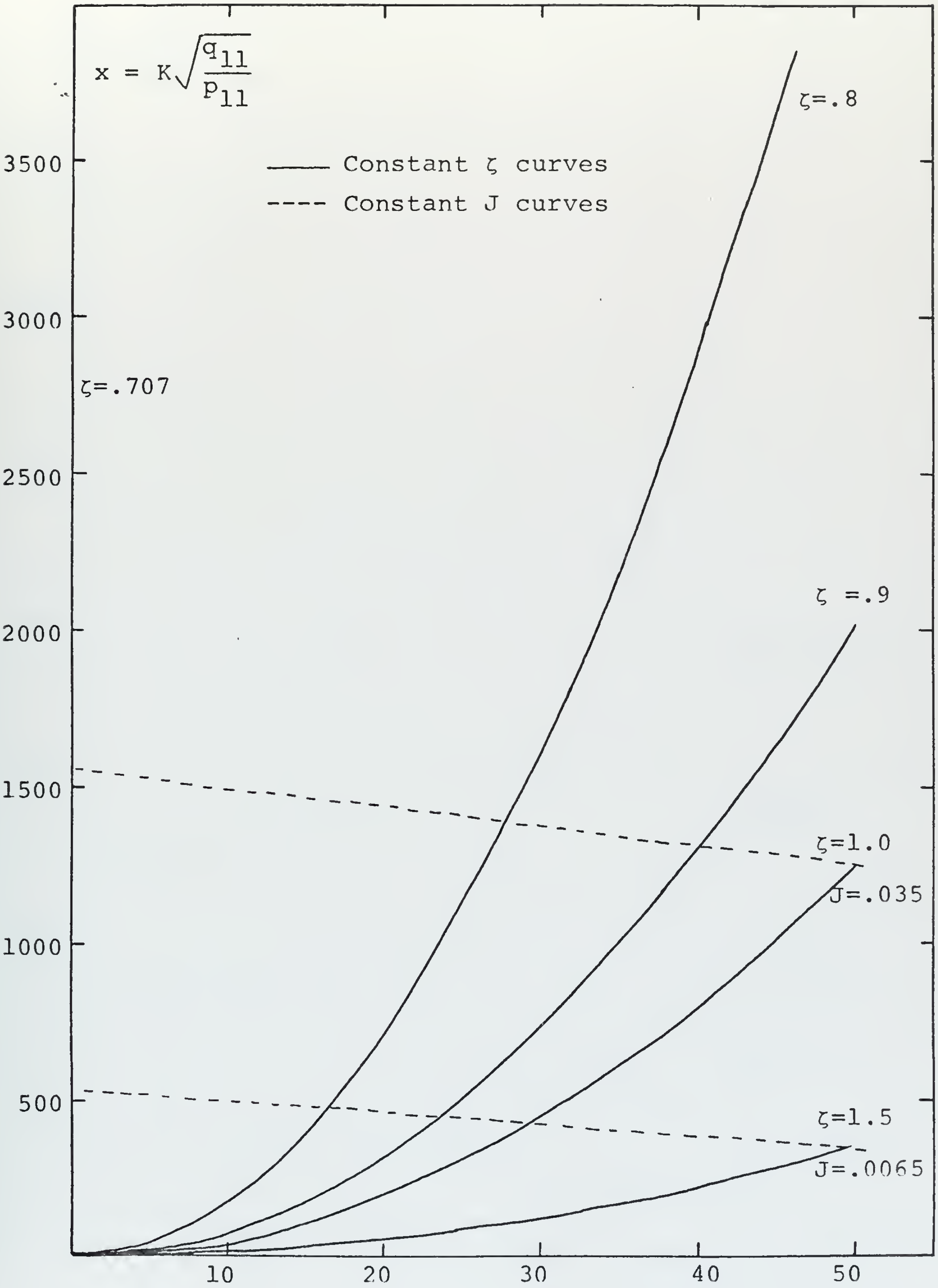


Figure 3.3a. Parameter plane optimal regulator when $W = \sqrt{\frac{q_{22}}{q_{11}}}$, negative tachometer feedback for J curves K assumed to 1600.

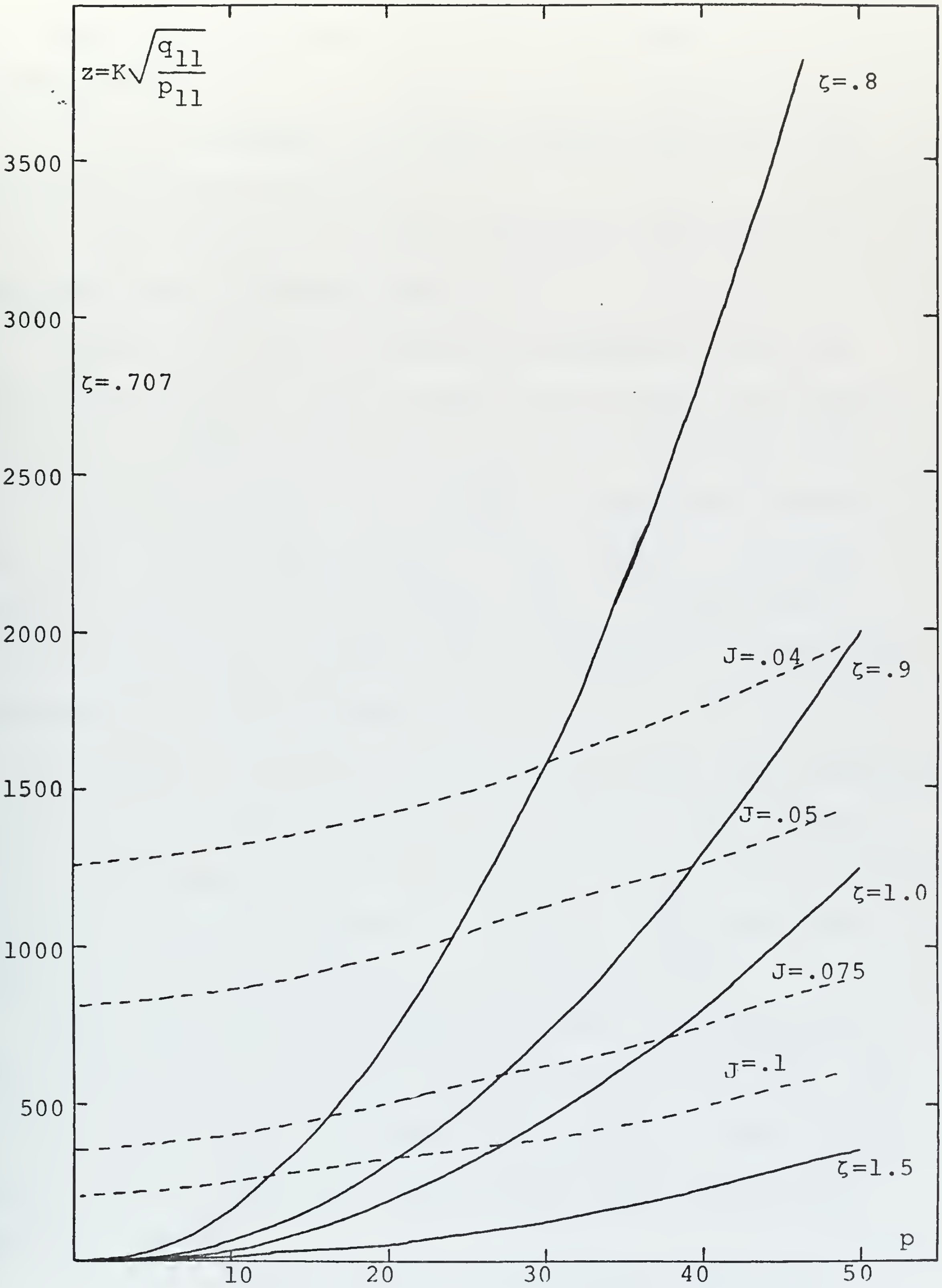


Figure 3.3b. Parameter plane optimal regulator when W=0 negative tachometer feedback for J curves $\sqrt{\frac{q_{11}}{p_{11}}}$ to be 1.

From these two figures the following results can be established:

1. For K constant, the cost function increases with increasing weighting factor.

2. For constant weighting factor the cost function decreases with increasing gain.

Figures 3.4a thru 3.4d show the parameter plane for $W \neq 0$ and it can be readily seen that for any p and ζ two values of ω_n can be obtained, both having different cost.

In Figure 3.4a and 3.4b, $\sqrt{\frac{q_{11}}{p_{11}}}$ has been kept constant at 1 but for the cost analysis a value of $W = 1$ has been used in 3.4a and $W = .5$ in Figure 3.4b. A comparison of these curves shows a larger cost for larger W, and the cost decreasing with an increase in gain.

On Figures 3.4c and 3.4d the gain has been kept constant at 10 and the weighting factor q_{11} varied. These figures show a decrease in cost for a decrease in q_{11}, and a slightly smaller cost for larger W. The value of p_{11} has been kept constant for these and the following cases.

When using positive feedback any value of damping can be used and the results are shown in Figures 3.5 thru 3.7d.

In Figure 3.5, K has been kept constant at 1600 and q_{11} varied and in Figure 3.6, q_{11} has been kept constant and the gain varied.

The loci of equal cost function follows different patterns in both figures but in general an increase in cost follows a decrease in gain.

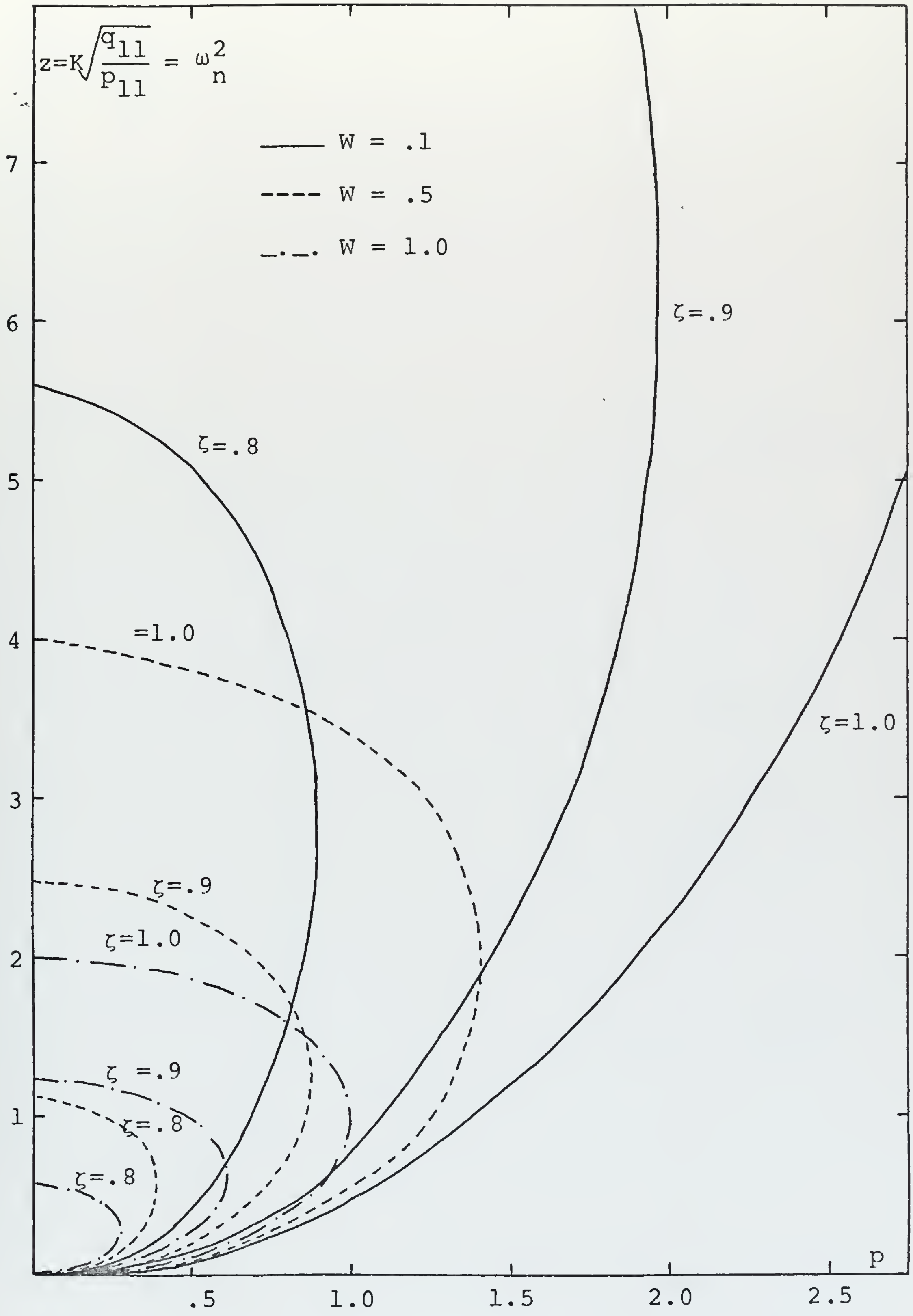


Figure 3.4. Constant ζ curves for different values of W, negative tachometer feedback.

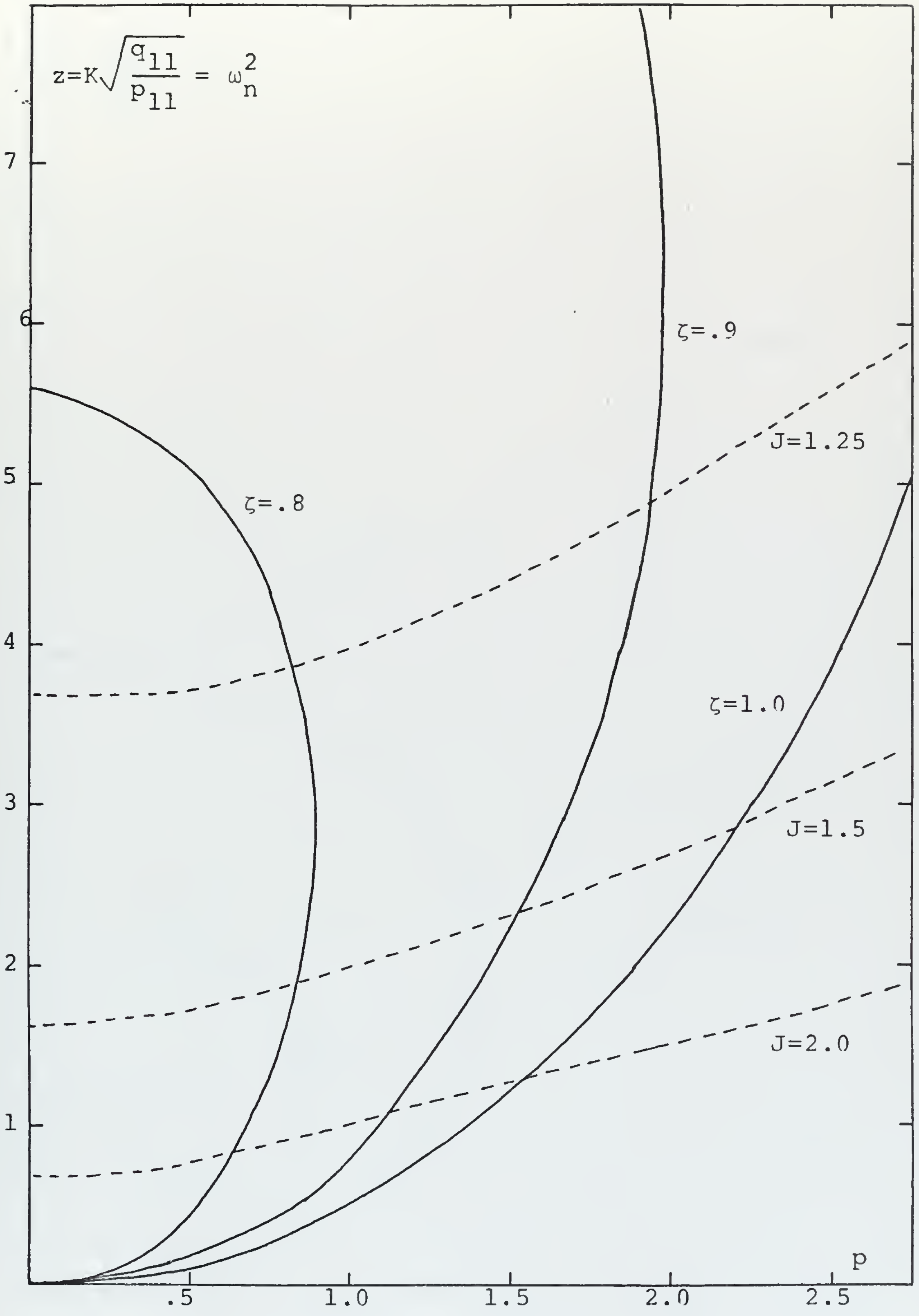


Figure 3.4a. Constant J curves for W = .1 and $\sqrt{\frac{q_{11}}{p_{11}}} = 1$, negative tachometer feedback.

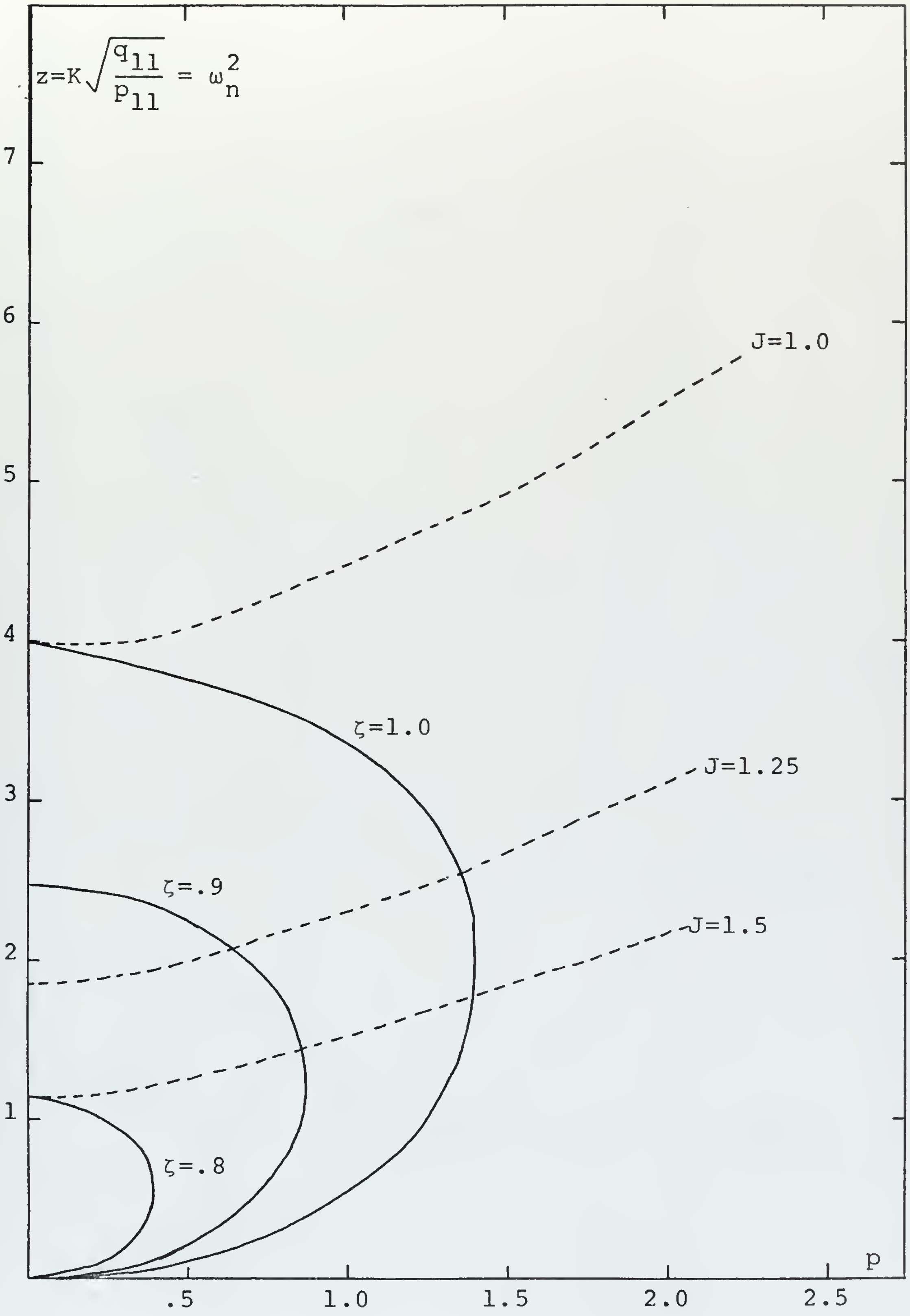


Figure 3.4b. Constant cost curves for W = .5 and $\sqrt{\frac{q_{11}}{p_{11}}} = 1$, negative tachometer feedback.

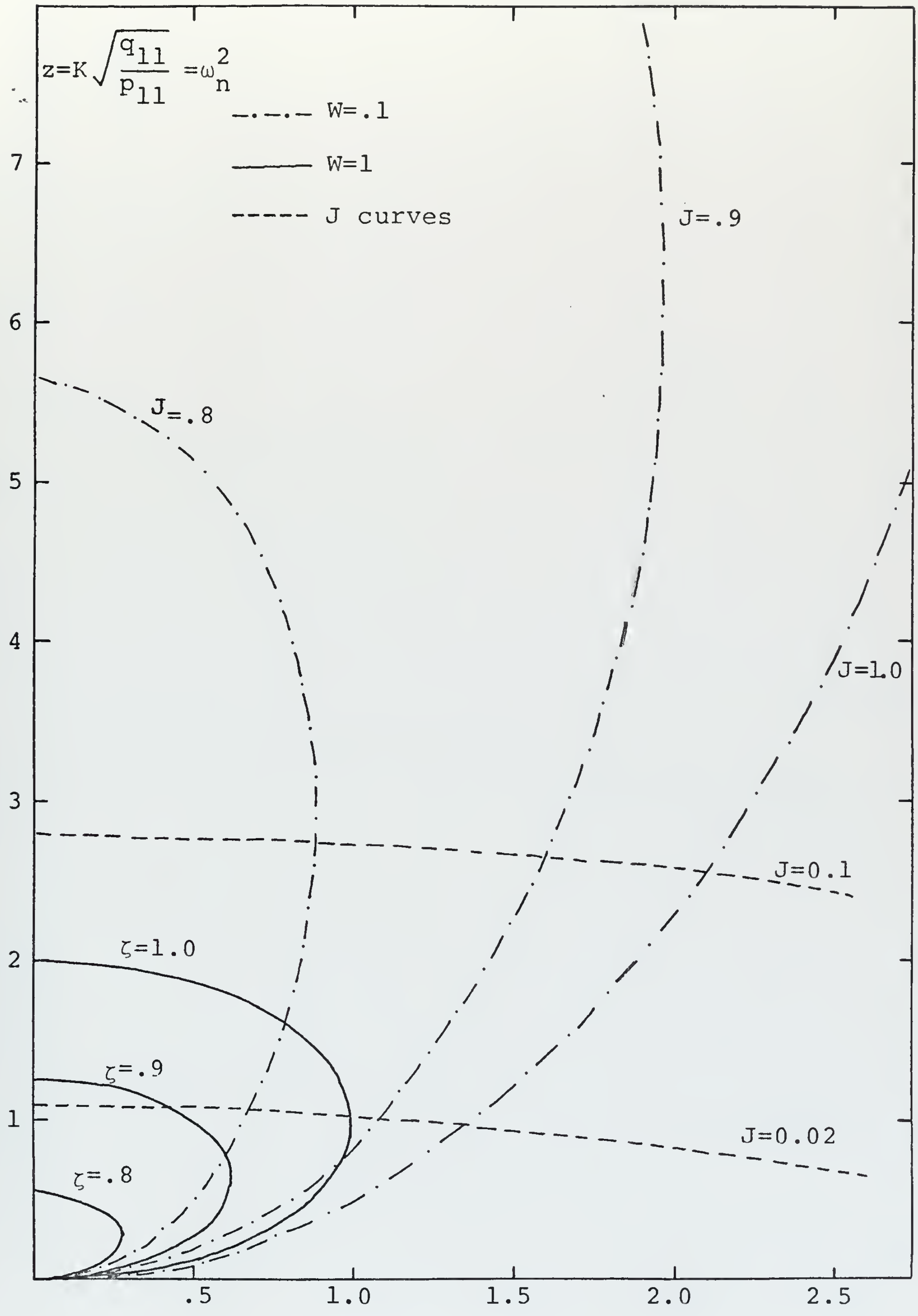


Figure 3.4c. Constant J curves for W = 1.0 and K = 10, negative tachometer feedback.

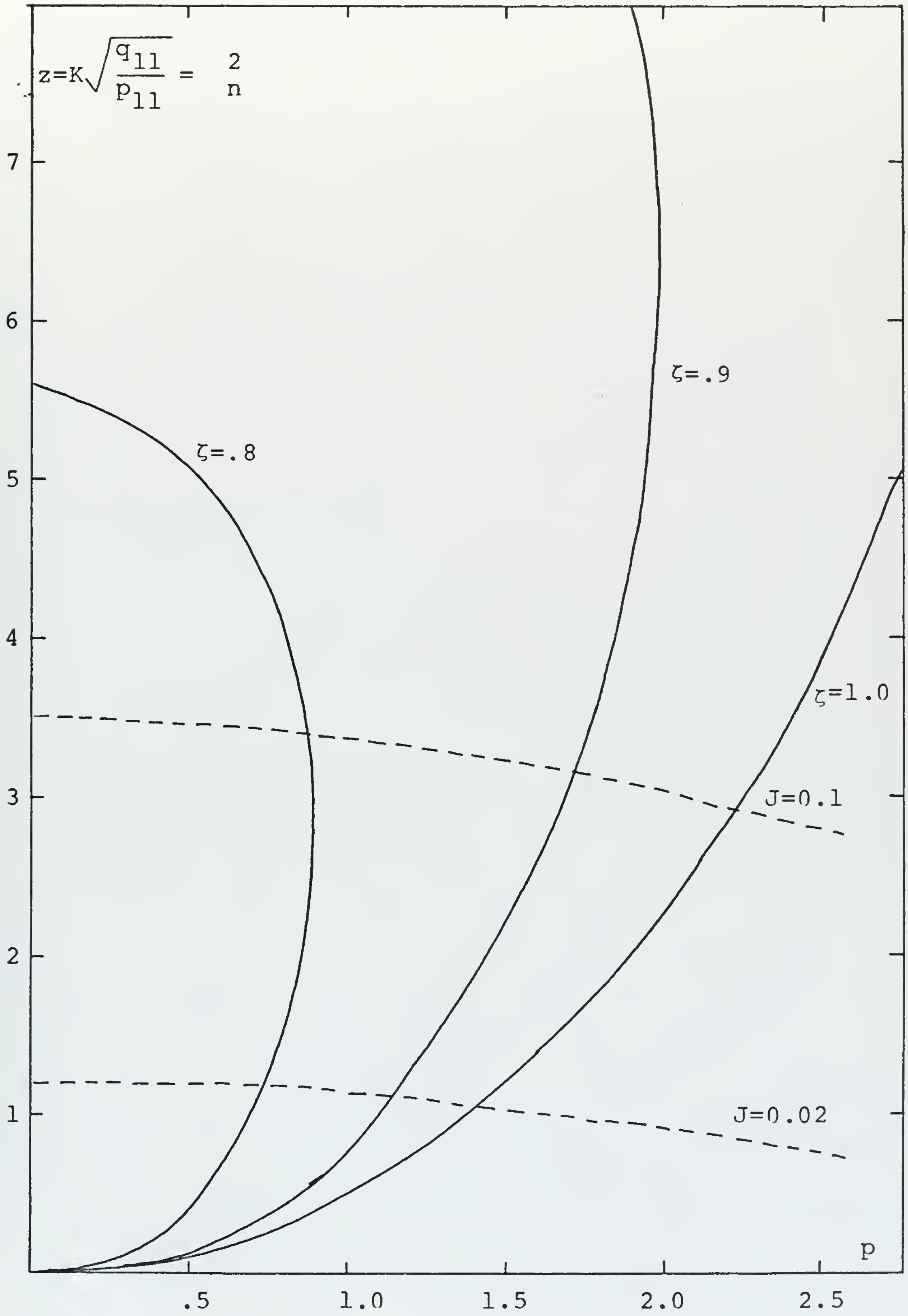


Figure 3.4d. Constant J curves for W = .1 and K = 10, negative tachometer feedback.

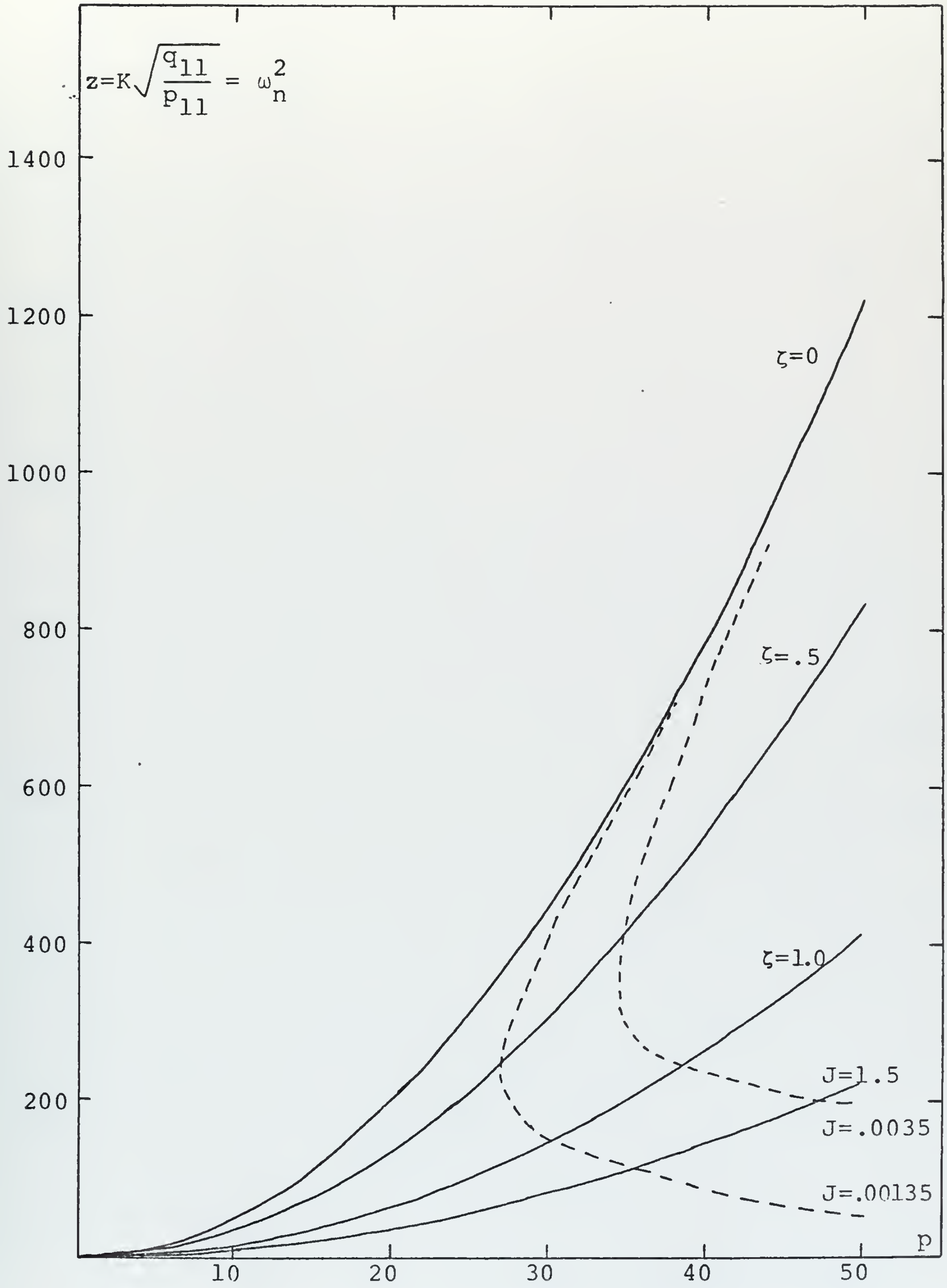


Figure 3.5. Constant ζ curves for W = 0 and positive feedback for constant J curves, K = 1600.

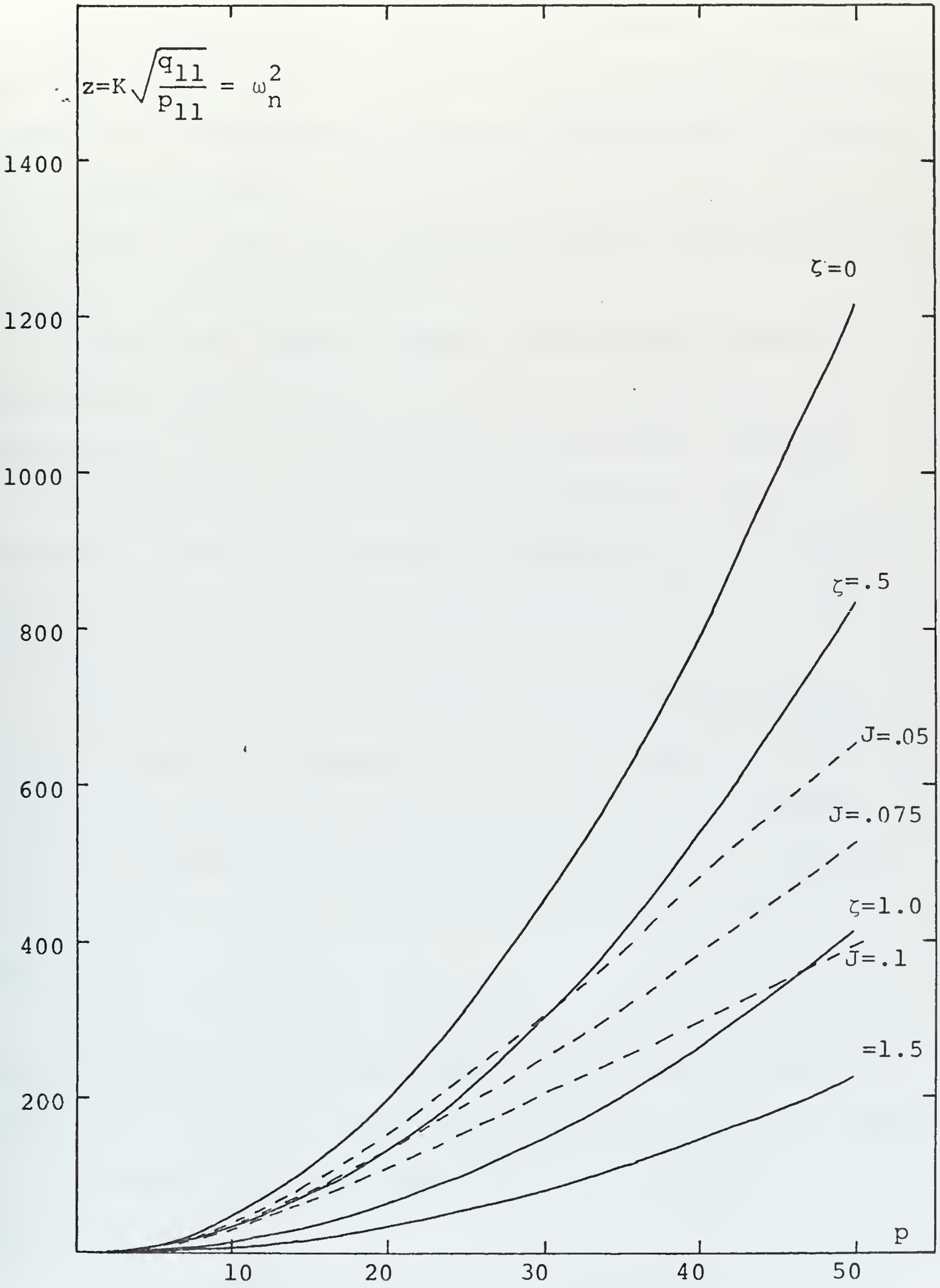


Figure 3.6. Constant curves for W = 0 and positive tachometer feedback for constant J curves, $\sqrt{\frac{q_{11}}{p_{11}}} = 1$.

A comparison between Figure 3.3 and Figure 3.5 shows that for negative feedback values of $\zeta \geq .707$ are restricted to the first quadrant while for positive feedback any value of ζ is permissible.

Figures 3.7 show the parameter plane for two different values of W and for different ζ.

A comparison between negative and positive feedback shows same results as for W = 0, i.e., any value of ζ is permissible in the first quadrant for positive feedback.

Figure 3.7a and 3.7b compare the effect on cost of different values of W, keeping q_{11} constant and varying the gain which results in larger cost, which for p = 1.5 is 0.75 for W = 1 and 1.25 for W = 2.

Figures 3.7c and 3.7d compare effect on cost of different values of W, keeping the gain constant at 10 and varying q_{11}. For a small value of p the cost is smaller for W = 1 than for W = 2 and for any ω_n there is a minimum cost, which for $\omega_n = .7$ is about 0.19 for W = 1 and 0.3 for W = 2.

Keeping $\zeta = 1.0$ and K constant at 10, a curve of cost against frequency has been drawn and the results shown in Figure 3.8. It can be seen that the cost follows the shape of a parabola increasing very fast as the frequency approaches 2.

In order to study the effect of the weighting factors on the steady state accuracy it is convenient to refer to the equivalent block diagram of Chapter I. It can be seen that

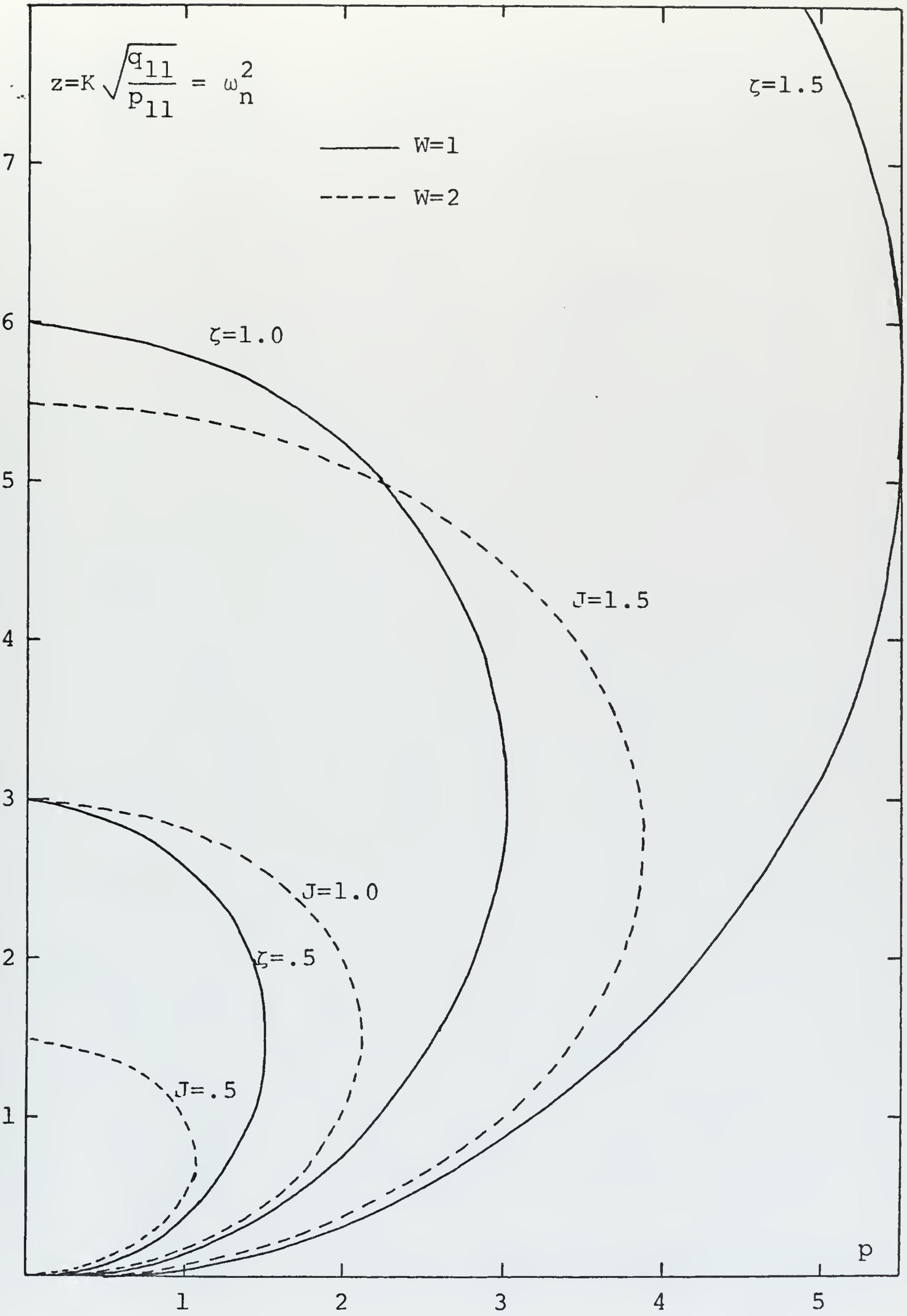


Figure 3.7. Constant J curves for W ≠ 0, positive tachometer feedback.

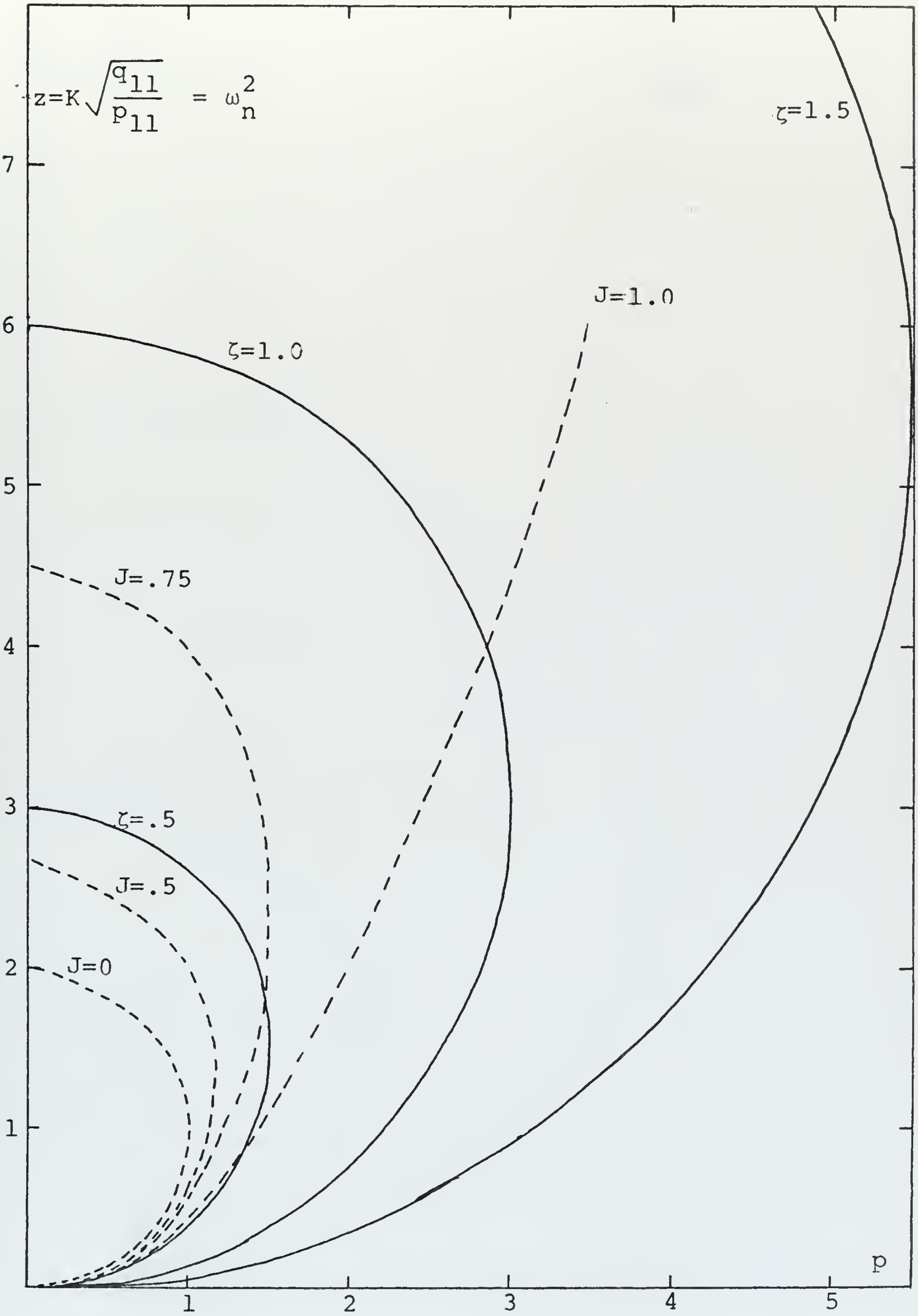


Figure 3.7a. Constant J curves for W=1 and $\sqrt{\frac{q_{11}}{p_{11}}}=1$, positive tachometer feedback.

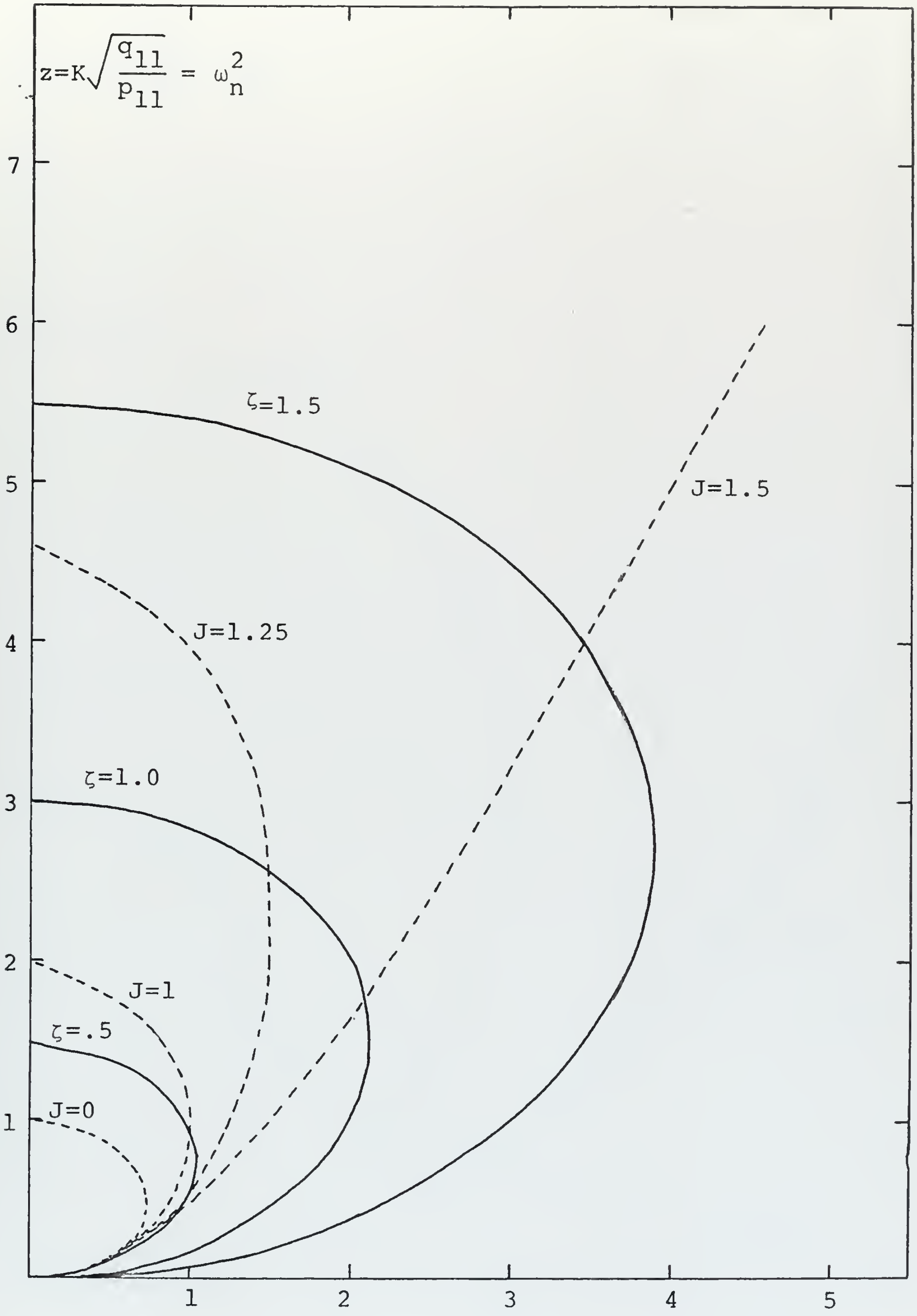


Figure 3.7b. Constant J curves for W=2 and $\sqrt{\frac{q_{11}}{p_{11}}}=1$, positive tachometer feedback.

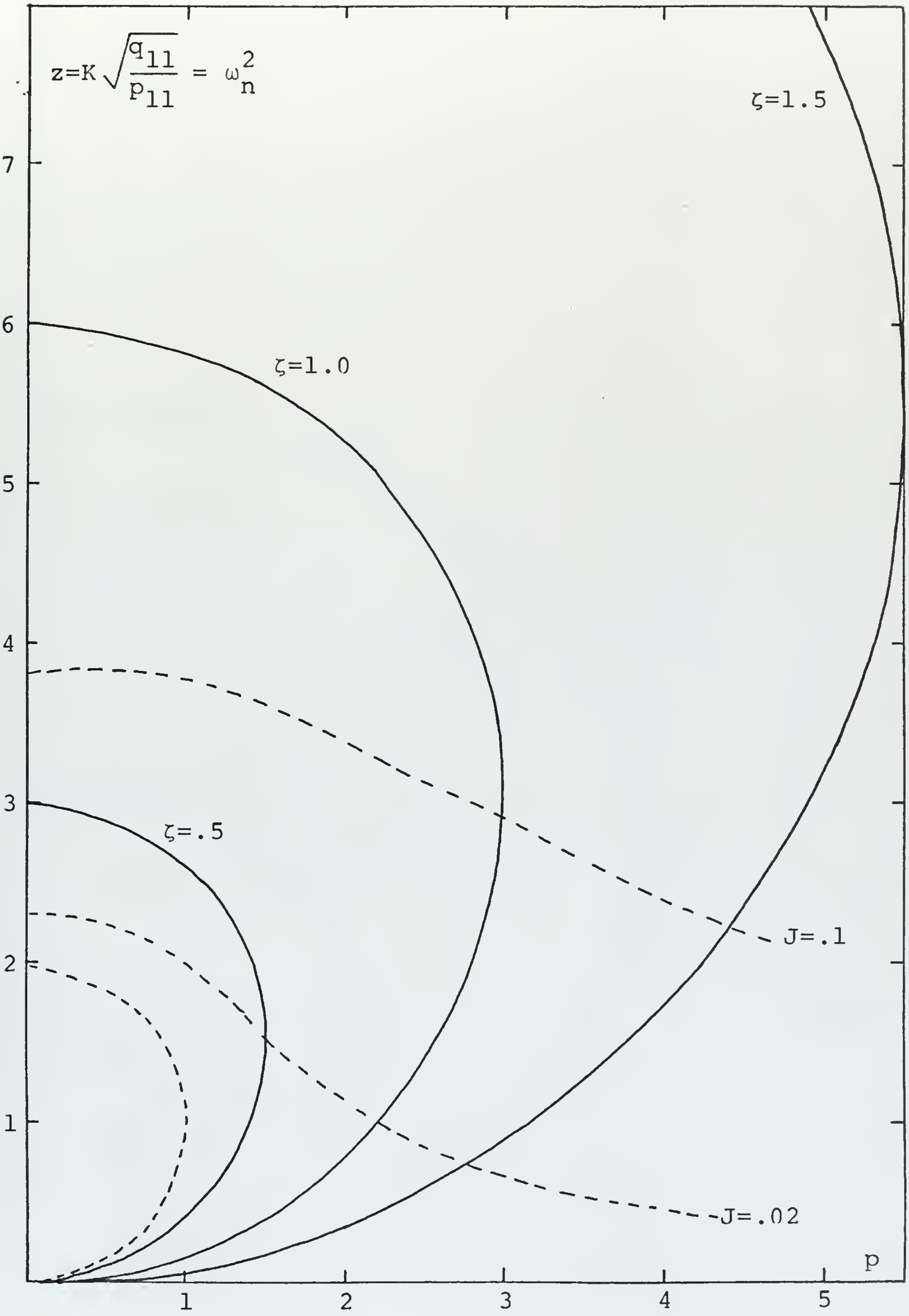


Figure 3.7c. Constant J curves for W = 1 and K = 10, positive tachometer feedback.

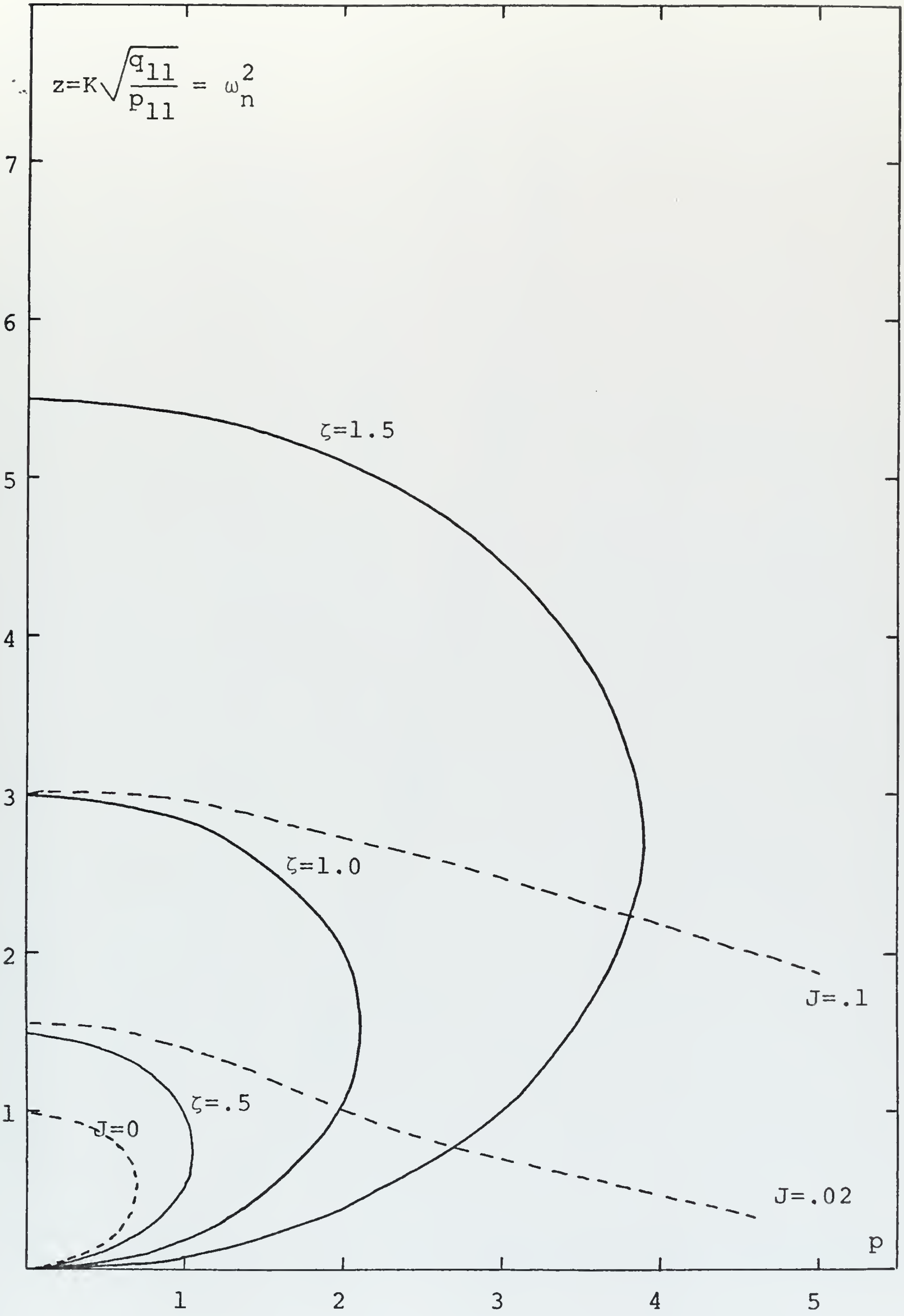


Figure 3.7d. Constant J curves for W=2 and K=10, positive tachometer feedback.

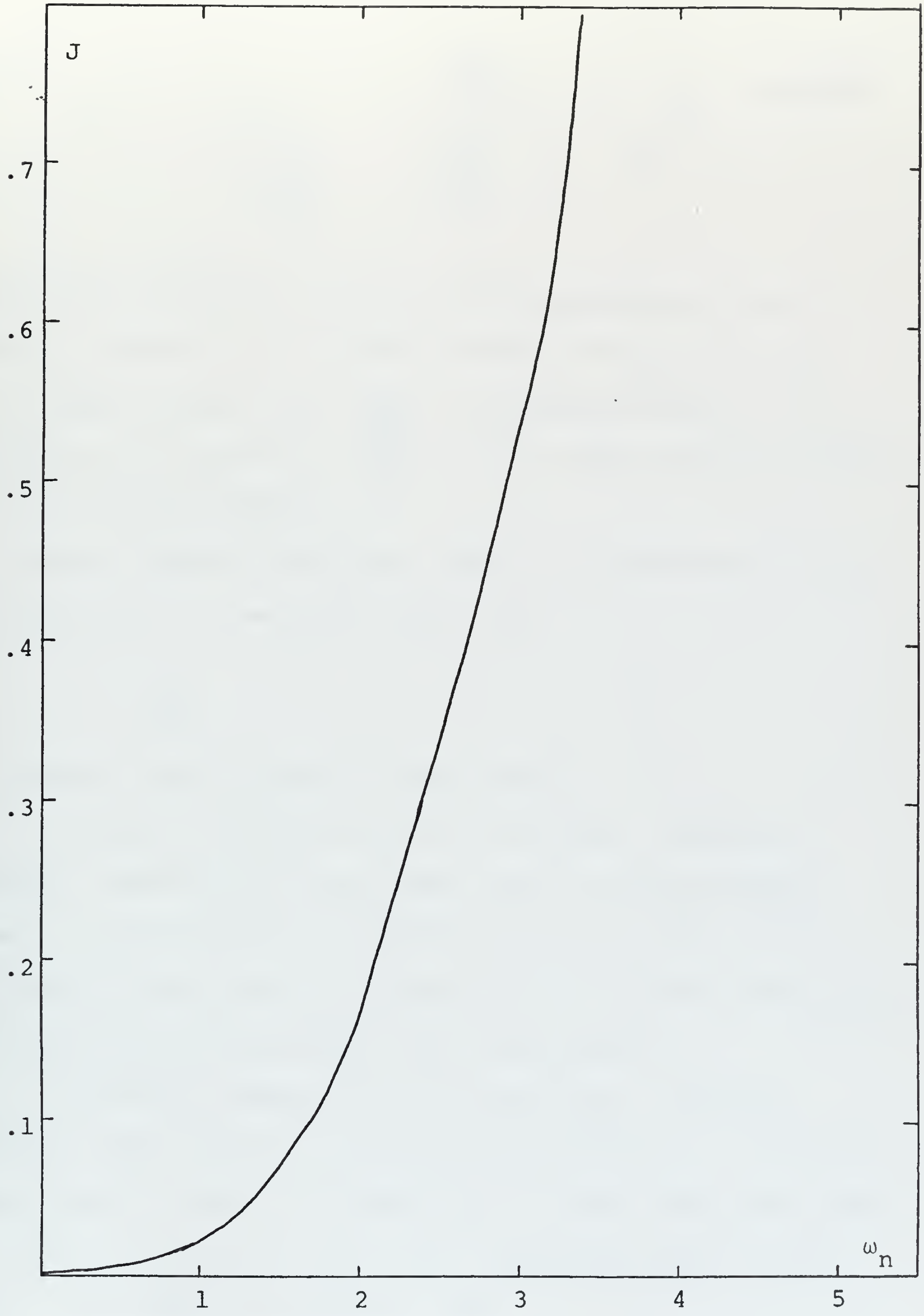


Figure 3.8. J-ω_n parameter plane for ζ=1.0 and K=10.

$$E_{ss} = \frac{1}{1 + \frac{1}{\sqrt{\frac{q_{11}}{p_{11}}}}} = \frac{\sqrt{\frac{q_{11}}{p_{11}}}}{\sqrt{\frac{q_{11}}{p_{11}}} + 1} = \frac{\frac{z}{K}}{\frac{z}{K} + 1}$$

which will depend only on q_{11} and the parameter plane is shown in Figure 3.9. For small values of q_{11} the steady state error is also very small, increasing rapidly as q_{11} increases to a value of $\sqrt{\frac{q_{11}}{p_{11}}} = 2$, and then slowly approaching 1.0 in the infinite.

For the speed of response study it is convenient to use the definition of settling time, i.e.,

$$\tau = \frac{4}{\zeta\omega_n}$$

Several cases should be considered:

1. Negative feedback for the velocity. Curves of constant settling time have been drawn in the parameter plane for two different weighting functions, W=0 and W=2, the results being shown on Figure 3.9a. It can be seen that the settling time decreases with increasing W.

2. Positive feedback for the velocity. Constant settling time curves for the same parameters as in the previous case are shown on Figure 3.9b. This curves show that the settling time decreases with increasing W but faster than in the negative feedback case.

3. Negative feedback for the position. The results are as in Case 1.

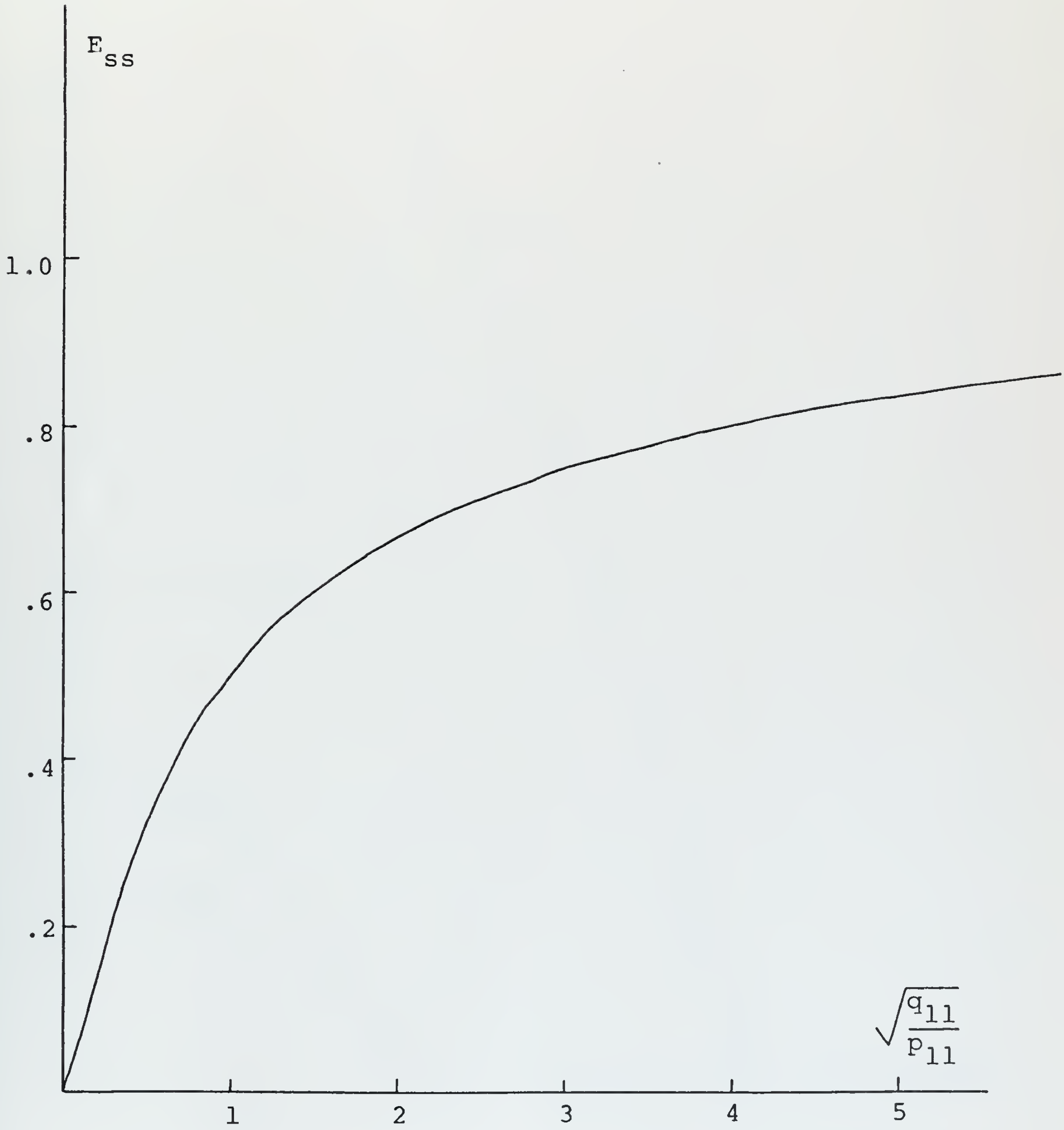


Figure 3.9. Steady state error vs. $\sqrt{\frac{q_{11}}{p_{11}}}$ parameter plane.

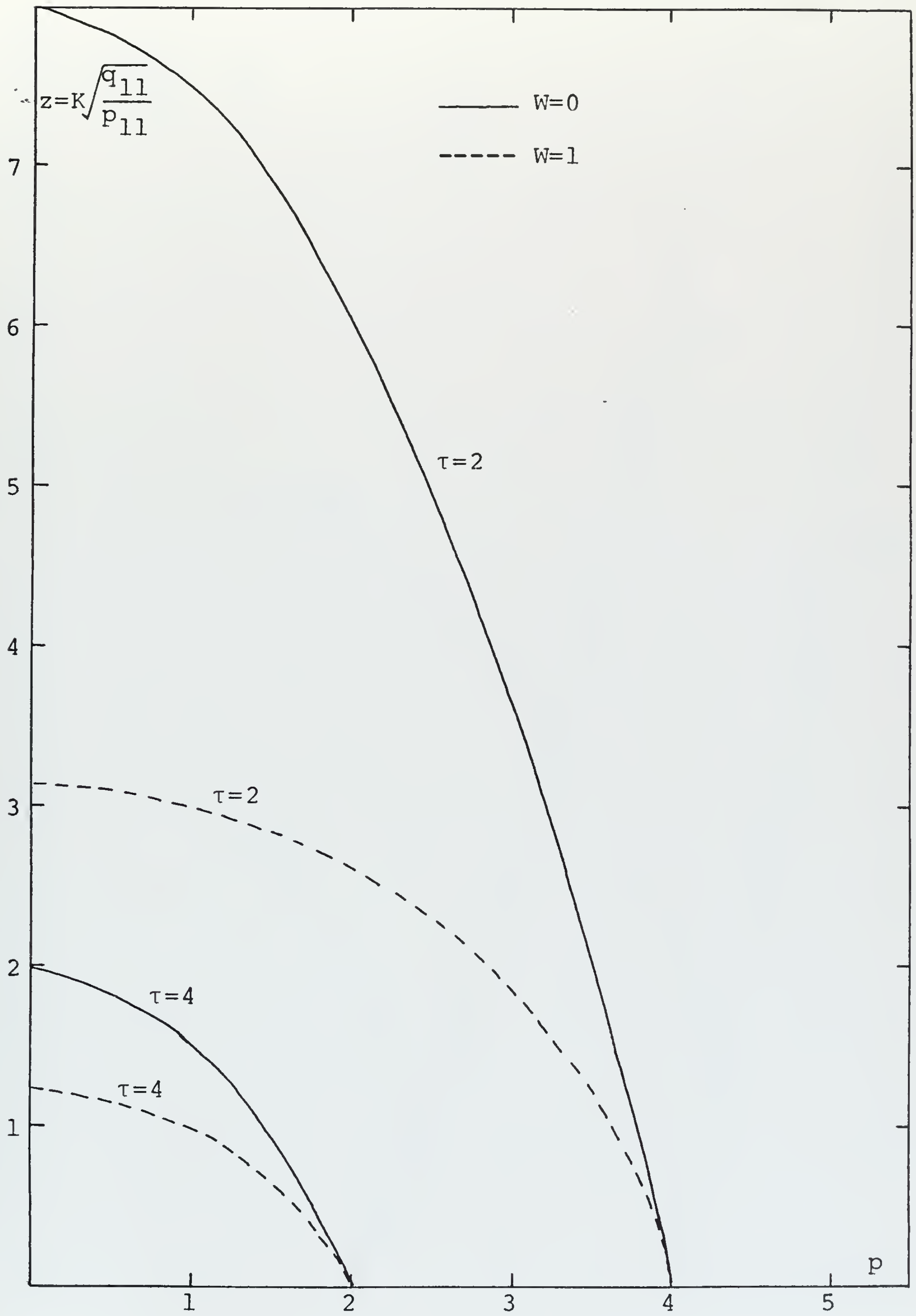


Figure 3.9a. Constant settling time curves for negative tachometer feedback.

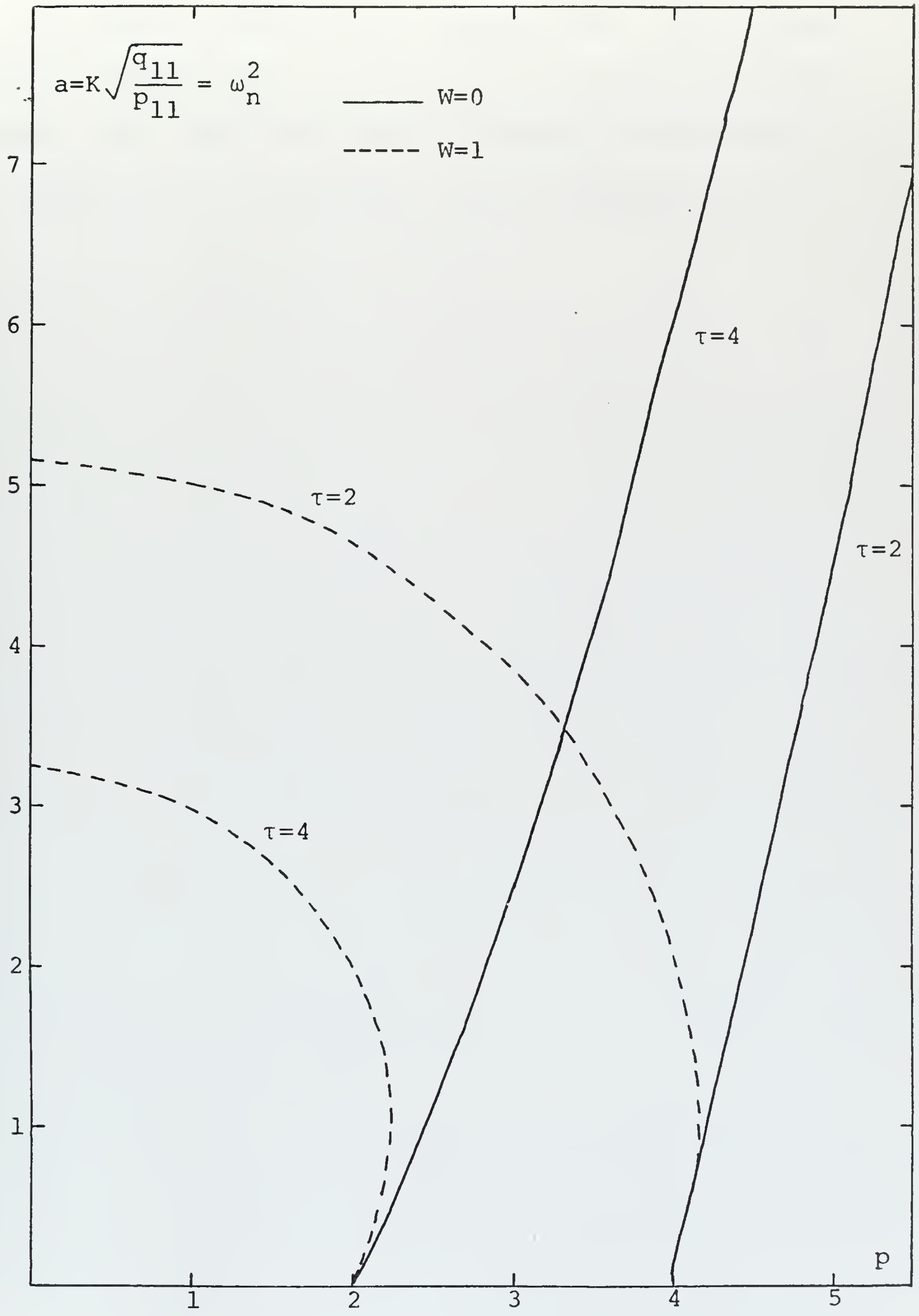


Figure 3.9b. Constant settling time curves for positive tachometer feedback.

4. Positive feedback for the position. Using Routh criteria of stability it can be seen that this case gives a pole in the right hand plane, meaning that the system will be unstable and the regulator will not regulate at all.

IV. SENSITIVITY IN THE OPTIMAL REGULATOR

A. GENERAL

An optimal design guarantees minimum cost for the regulator from any initial conditions.

If any parameter of the regulator deviates from the value used in (or determined by) the optimal design, then the cost of regulating is increased, i.e., the performance of the system is not optimal.

Sensitivity is a word used to describe the rate at which some characteristics of the system deviates from a reference value as a function of a parameter change. Thus various types of sensitivity could be defined:

a. Root sensitivity

b. Bandwidth sensitivity

c. Steady state accuracy sensitivity

d. Settling time sensitivity

e. Rise time sensitivity

f. State sensitivity

g. Cost sensitivity

State sensitivity is a measure of the deviation of a dynamic state from the values it would assume if the system was optimal.

Cost sensitivity is some measure of the deviation of the cost from the optimal cost.

Ultimately all of these various sensitivities relate to the same basic characteristics of a linear regulator, i.e.,

they indicate changes in root location (possibly excepting steady state accuracy sensitivity). However there must be some basic differences in these various sensitivities in the sense that some are vector quantities and others are scalars, i.e., root, bandwidth, settling time and rise time sensitivities all indicate pretty clearly the direction in which a root has moved, while cost sensitivity only indicates the magnitude of the root motion.

From a design point of view there may be some advantages to the cost sensitivity, i.e., one can usually accept dominant root location within a specified s-plane area, and a defined area on the parameter plane normally maps into a dominant area on the s-plane. Correlation with other performance criteria may very well be required.

Cost may be evaluated at any point on the parameter plane and thus constant cost contours can be obtained. Cost sensitivity in a macroscopic sense is then just the difference between two costs divided by the increment in parameter value between them.

Cost sensitivity can also be evaluated at a point and normally the point of greatest interest on the parameter plane would be the point at which the parameter assume optimal values. Change in cost from this optimal value (for a small change) can be computed by:

$$\frac{J}{\partial(\text{parameter})} \times \Delta\text{parameter} = \Delta J$$

$$J_{\text{opt}} + \Delta J = J_{\text{at new point}}$$

This however depends on initial conditions values, and the permissible range of Δparameter depends on the system and on the cost function.

B. SENSITIVITY OF THE COST FUNCTION AND SENSITIVITY OF THE OPTIMAL COST FUNCTION

In the research, the curves of constant J on the parameter plane are obtained using an expression for J which is derived in terms of p, K, K_1, K_2, where p and K are plant parameters and K_1 and K_2 are values for the state feedback loops. In order to use the expressions the system is optimized at some chosen p and K, and values of K_1 and K_2 are computed for the system thus optimized. When J is set to a non-optimal value K_1 and K_2 remain fixed at the optimal values and p and K are computed. The system is not optimal for the new p and K.

Sensitivity of the optimal system could be defined by taking the derivative of the cost function, i.e.,

$$S^J_{op} = \left.\frac{\partial J_o}{\partial p}\right|_{p=p_o,\ K=K_o}$$

$$S^J_{oK} = \left.\frac{J_o}{\partial K}\right|_{p=p_o,\ K=K_o}$$

However the equation for J optimal can be written in at least two forms:

A. The form is used to compute J curves, which contains the feedback gains K_1 and K_2 as constants.

B. A form obtained directly from the R matrix such that K_1 and K_2 do not appear as symbols, the cost being

expressed entirely in terms of p, K, q_{11}, q_{22}, p_{11}.

Both forms A and B should give the same numerical answer when $p = p_{opt}$ and $K = K_{opt}$. However their derivatives at the optimal point should be different, because form A describes a system that is not optimal when p and K are changed from their optimal values, while form B presumably expresses the condition whereby changes in p and K automatically result in changes in K_1 and K_2 so that the system remains optimal at the new p, K values, but of course with different value of J.

Under these conditions the choice of a definition for sensitivity becomes a matter of concern:

Should sensitivity be defined on the basis of $|J|$ from the value it has when p and K are at their specified optimal values? or should be defined on the basis of the deviation of $|J|$ at point p, K, from $|J_{opt}|$ at point p,K?

By this last statement it is meant:

If K_1 and K_2 are evaluated at p, K, then

$$J_o = f(p_o, K_o, K_1, K_2)$$

At some new set of plant parameters values p_a, K_a,

$$J_1 = f(p_a, K_a, K_1, K_2)$$

which corresponds to the curves drawn on the parameter plane, but the system is optimal at p_a, K_a. If the system were optimal at p_a, K_a, K_{1a}, K_{2a}, would have to be determined and the cost function would become

$$J_{1a} = f(p_a, K_a, K_{1a}, K_{2a}) \quad \text{optimal.}$$

C. SENSITIVITY OF COST FUNCTION TO PLANT PARAMETERS VARIATIONS

If K_1 is adjusted for the optimal value, then

$$K_1 = \sqrt{\frac{q_{11}}{p_{11}}}$$

and

$$\frac{J}{p_{11}} = \left\{ 2K_1^2 + \frac{2K_1}{K} (p + K_2K)^2 + K_1K(K_2^2 + \frac{q_{22}}{q_{11}}) \right.$$

$$\left. -2K_1K_2(p + K_2K) \right\} \frac{x_1^2(0)}{2(p + K_2K)}$$

$$+ \frac{2K_1}{K} x_1(0)\ x_2(0)$$

$$+ \left\{ \frac{2K_1}{K} + K_2^2 + \frac{q_{22}}{p_{11}} \right\} \frac{x_2^2(0)}{2(p + K_2K)}$$

If K_2 is also adjusted for the optimal value then

$$\frac{q_{22}}{p_{11}} = K_2^2 + \frac{2K\ p_o}{K_o} - \frac{2K_1}{K_o}$$

and

$$\frac{J}{p_{11}} = \left\{ 2K_1^2 + \frac{2K_1}{K} (p + K_2K)^2 + K_1K(2K_2^2 + \frac{2K\ p_o}{K_o} - \frac{2K_1}{K_o}) \right.$$

$$\left. -2K_1K_2(p + K_2K) \right\} \frac{x_1^2(0)}{2(p + K_2K)}$$

$$+ \frac{2K_1}{K} x_1(0)x_2(0)$$

$$+ \left\{ \frac{2K_1}{K} + 2K_2^2 + \frac{2K_2p_o}{K_o} - \frac{2K_1}{K_o} \right\} \frac{x_2^2(0)}{2(p + K\ K)}$$

This is the cost function assuming that K_1 and K_2 are adjusted to optimum for the nominal values of system parameters, p_o and K_o. This result may be illustrated more graphically by rearranging as follows:

$$\frac{J}{p_{11}} = \left\{ 2K_1^2 \left(1 - \frac{K}{K_o} \right) + 2K_1K_2 \left(\frac{K_1}{K_o} p_o - p \right) \right.$$

$$\left. + \frac{2K_1}{K} (p + K_2K)^2 \right\} \frac{x_1^2(0)}{2(p + K_2K)}$$

$$+ \frac{2K_1}{K} x_1(0) \; x_2(0)$$

$$+ \left\{ 2K_1 \left(\frac{1}{K} - \frac{1}{K_o} \right) + \frac{2K_2(p_o + K_2K_o)}{K_o} \right\} \frac{x_2^2(0)}{2(p + K_2K)}$$

The sensitivity becomes:

$$\frac{J - J_{opt}}{p_{11}} = \left\{ \frac{K_1^2 \left(1 - \frac{K}{K_o}\right) + K_1K_2 \left(\frac{Kp_o - pK_o}{K_o} \right)}{p + K_2K} \right.$$

$$\left. + \frac{K_1}{K} (p + K_2K) - \frac{K_1}{K} (p_o + K_2K_o) \right\} x_1^2(0)$$

$$+ 2 \left[\frac{K_1}{K} - \frac{K_1}{K} \right] x_1(0) x_2(0)$$

$$+ \left\{ \frac{K_1 \left(\frac{1}{K} - \frac{1}{K_o} \right)}{p + K_2K} + \frac{K_2}{K} \left[\frac{p_o + K_2K_o}{p + K_2K} \right] - 1 \right\} x^2(0) \qquad (4.1)$$

$$\frac{J - J_{opt}}{p_{11}} = \left\{ \frac{\frac{K_1^2}{K_o} (K_o - K) + \frac{K_1K_2}{K_o} (Kp_o - pK_o)}{p + K_2K} + K_1 \left(\frac{p}{K} - \frac{p_o}{K_o} \right) \right\} x_1^2(0)$$

$$+ 2K_1\left[\frac{1}{K} - \frac{1}{K_o}\right]x_1(0)x_2(0)$$

$$+ \left\{\frac{K_1\left(\frac{K_o - K}{KK_o}\right)}{p + K_2K} + \frac{K_2(p_o - p) + K_2(K_o - K)}{K_o(p+K_2K)}\right\} x_2^2(0)$$

Note these become zero when $K \rightarrow K_o$, $p \rightarrow p_o$

$$\frac{J - J_{opt}}{p_{11}} = \left\{\left(\frac{K_1^2}{K_o(p+K_2K)}\right)(K_o-K) + \left(\frac{K_1K_2}{K_o(p+K_2K)} - \frac{K_1}{KK_o}\right)(Kp_o-pK_o)\right\}x_1^2(0)$$

$$+ \frac{2K_1}{KK_o}(K_o-K)\ x_1(0)x_2(0)$$

$$+ \left\{\left(\frac{K_1}{KK_o} + \frac{K_2^2}{K_o}\right)(K_o-K) + \frac{K_2}{K_o}(p_o-p)\right\}\frac{x_2^2(0)}{(p+K_2K)}$$

For small changes where

$$K = K_o + \delta K \quad ; \quad K \simeq K_o$$

$$p = p_o + \delta_p \quad ; \quad p = p_o$$

$$\frac{\delta J}{p_{11}} = \left\{\left(\frac{K_1^2}{K_o + (p_o + K_2K_o)}\right)(1-\delta K)\right.$$

$$\left. + \frac{1}{K_o}\left(\frac{K_1K_2}{p_o+K_2K_o} - \frac{K_1}{K_o}\right)\left[(\delta K)p_o - K_o(\delta p)\right]\right\} x_1^2(0)$$

$$- \frac{2K_1}{K_o^2}(\delta K)x_1(0)x_2(0)$$

$$+ \left\{\left(\frac{K_1}{K_o^2} + \frac{K_2}{K_o}\right)(-\delta K) + \frac{K_2}{K_o}(-\delta p)\right\}\frac{x_2^2(0)}{p_o+K_2K}$$

$$\frac{\delta J}{p_{11}} = \left\{\frac{-\delta K}{K_o^2}\left[\frac{p_o^2K_1+K_1^2K_o}{p_o+K_2K_o} - \frac{K_1p_o+K_1K_2}{K_o}\right] - \delta p\left[\frac{p_oK_1}{K_o(p_o+K_2K_o)}\right]\right\}x_1^2(0)$$

$$- \frac{2K_1}{K_o^2} (\delta K) x_1(0) x_2(0)$$

$$+ \left\{ - \frac{K_1+K_2K_o}{K_o^2(p_o+K_2K_o)} (\delta K) - \frac{K_2}{K_o+(p_o+K_2K_o)} (\delta p) \right\} x_2^2(0)$$

ACKNOWLEDGEMENT:

The derivation of this expression was originally worked by Dr. Sydney R. Parker.

From 4.1 if $p = p_o = 30$ and K deviates from the optimal value of 1600 a curve can be represented in the parameter plane and is shown in Figure 4.1. Also shown is the curve for $K = K_o = 1600$ and p deviating from the optimal value.

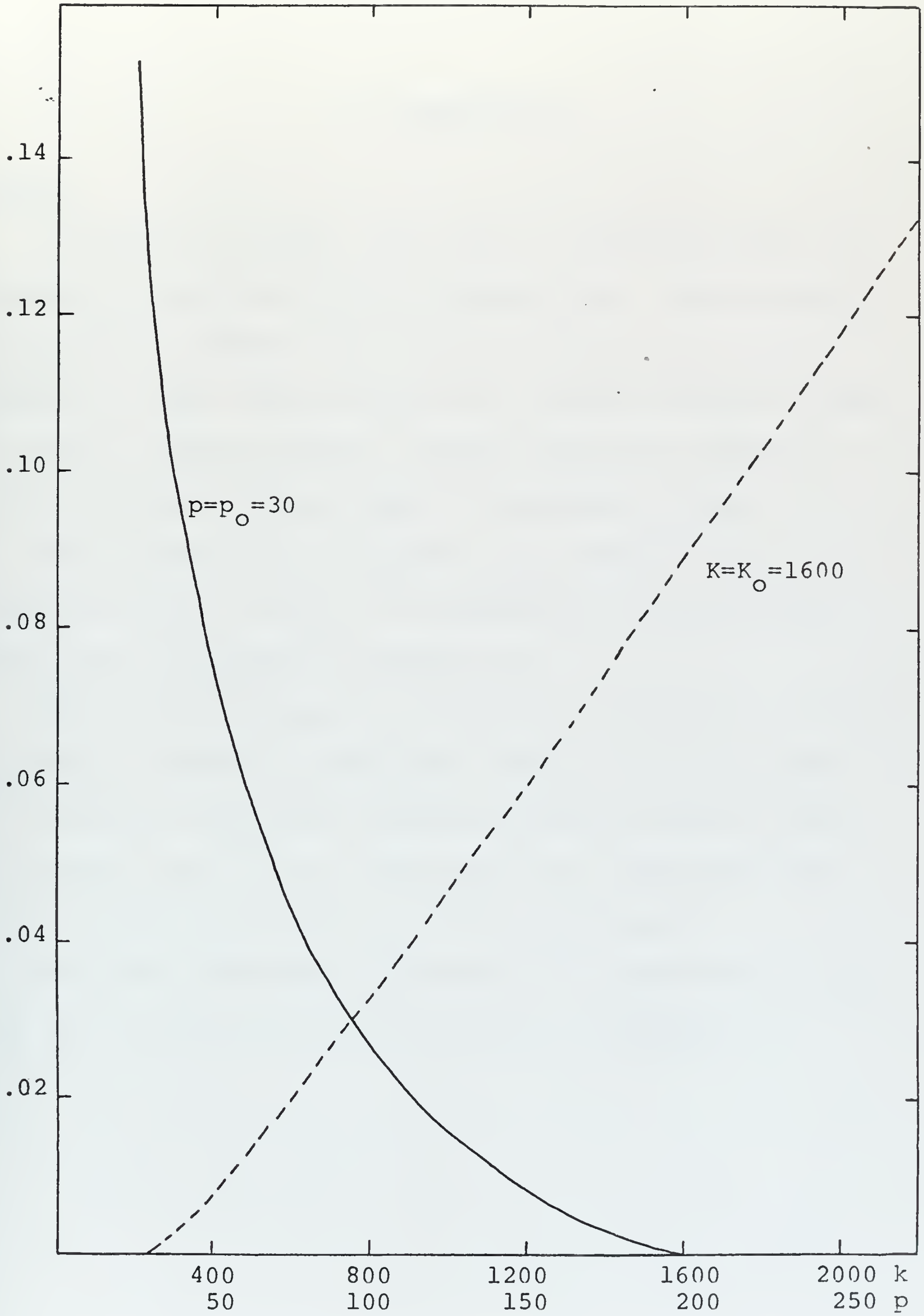


Figure 4.1. Sensitivity of cost function to plant parameter variation.

V. CONCLUSIONS

RESULTS

An optimal regulator can always be obtained using the parameter plane method for any given plant. Both negative and positive feedback can be used to achieve the desired results but only negative position feedback should be used to obtain a stable system. Positive position feedback gives an unstable system with no regulation at all.

For a given plant an optimal cost function can be obtained and the values of the feedback path gain calculated using the curves of Chapter II.

When a given frequency and damping is desired, the plant parameters needed to achieve the results can be obtained for different weighting functions with the use of figures of Chapter III, or for a fixed plant the performance and cost be obtained by the use of the same figures.

For plant parameters variations the sensitivity of the cost can be obtained using the figure in Chapter IV.

APPENDIX A

GENERAL EXPRESSION FOR THE R MATRIX

From the reduced Ricatti equation:

$$A^T R_o + R_o A - R_o B P^{-1} B^T R_o + Q = 0 \tag{A.1}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -p \end{bmatrix}, \quad A^T = \begin{bmatrix} 0 & 0 \\ 1 & -p \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ K \end{bmatrix}, \quad B^T = [0 \quad K]$$

$$R_o = \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix}$$

Substituting into A.1

$$\begin{bmatrix} 0 & 0 \\ 1 & -p \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} + \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -p \end{bmatrix} - \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} \begin{bmatrix} 0 \\ K \end{bmatrix} \begin{bmatrix} 0 & K \end{bmatrix}$$

$$\begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} + \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 \\ r_{11}-pr_{22} & r_{12}-pr_{22} \end{bmatrix} + \begin{bmatrix} 0 & r_{11}-pr_{12} \\ 0 & r_{12}-pr_{22} \end{bmatrix} - \begin{bmatrix} K^2 r_{12} & K^2 r_{12} r_{22} \\ K^2 r_{12} & K^2 r_{22} \end{bmatrix}$$

$$+ \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix} = 0$$

$$-K^2 r_{12}^2 + q_{11} = 0 \tag{A.2}$$

$$r_{11}-pr_{12}-K^2 r_{12} r_{22} = 0 \tag{A.3}$$

$$r_{11}-pr_{12}-K^2r_{12}r_{22} = 0 \tag{A.4}$$

$$2(r_{12}-pr_{22})-K^2r_{22}^2+q_{22} = 0 \tag{A.5}$$

From A.2

$$r_{12}^2 = \frac{q_{11}}{K^2} \quad , \quad r_{12} = \pm \frac{\sqrt{q_{11}}}{K}$$

Substituting in A.5

$$\frac{2\sqrt{q_{11}}}{K} - 2pr_{22} - K^2r_{22}^2 + q_{22} = 0$$

Since Q must be positive definite q_{11} must be positive, but both signs will be used with the radical.

$$K^2r_{22}^2 + 2pr_{22} - q_{22} \pm \frac{2\sqrt{q_{11}}}{K} = 0$$

$$r_{22} = \frac{-2p \pm \sqrt{4p^2+4K^2(q_{22} \pm \frac{2\sqrt{q_{11}}}{K})}}{2K^2}$$

$$= \frac{-p}{K^2} \pm \frac{\sqrt{4p^2 + 4K^2(q_{22} \pm \frac{2\sqrt{q_{11}}}{K})}}{2K^2}$$

As r_{22} must be positive only the positive sign outside the radical can be used, and the negative sign inside the radical only if

$$4K^2(q_{22} - \frac{2\sqrt{q_{11}}}{K}) \geq 0$$

i.e. for $\quad |q_{22}| \geq \left|\frac{2\ q_{11}}{K}\right|$

Then, both signs will be carried with the indicated restriction implied.

From A.3 and A.4

$$r_{11} = \frac{p\sqrt{q_{11}}}{K} + \left(\frac{\pm\sqrt{q_{11}}}{K}\right)\left(-p + \sqrt{p^2+K^2q_{22} \pm 2K\sqrt{q_{11}}}\right)$$

$$= \pm\frac{\sqrt{q_{11}}}{K}\sqrt{p^2+K^2q_{22} \pm 2K\sqrt{q_{11}}}$$

Thus the general expression for the R matrix is:

$$R_o = \begin{bmatrix} \pm\frac{\sqrt{q_{11}}}{K}\sqrt{p^2+K^2q_{22} \pm 2K\sqrt{q_{11}}} & \pm\frac{\sqrt{q_{11}}}{K} \\ \pm\frac{\sqrt{q_{11}}}{K} & \frac{-p}{K^2} + \sqrt{\frac{p^2}{K^4} + \frac{q_{22}}{K^2} \pm \frac{2\sqrt{q_{11}}}{K^3}} \end{bmatrix}$$

ACKNOWLEDGEMENT:

The derivation of this expression was originally worked by Dr. George J. Thaler.

APPENDIX B

GENERAL COST FUNCTION FOR A SECOND ORDER SYSTEM

The following derivation was originally worked by Dr. Sidney R. Parker and has been included because its extensive use in this thesis and not being published in any paper.

It has been shown that for the general second order regulator

$$J = \int_0^\infty (x^T Qx + U^T pU)\, dt$$

where

$$Q = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix} \; ; \quad p = p_{11} \text{ and } U = \begin{bmatrix} K_1 K_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

If

$$X^T Qx + U^T pU = x^T Mx \tag{B.1}$$

Then

$$x^T Qx + U^T pU = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$+ \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} p_{11}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \left\{ \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix} + \begin{bmatrix} K_1^2 & K_1 K_2 \\ K_1 K_2 & K_2 \end{bmatrix} p_{11} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} q_{11} + K_1^2 p_{11} & p_{11} K_1 K_2 \\ p_{11} K_1 K_2 & q_{22} + K_2^2 p_{11} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Then from B.1

$$m_{11} = q_{11} + K_1^2 p_{11}$$

$$m_{12} = m_{21} = p_{11} K_1 K_2$$

$$m_{22} = q_{22} + K_2^2 p_{11}$$

Then:

$$X^T MX = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= m_{11} x_1^2 + 2m_{12} x_1 x_2 + m_{22} x_2^2$$

If:

$$x_1 = \alpha_1 e^{-\lambda_1 t} + \alpha_2 \, e^{-\lambda_2 t}$$

then

$$x_2 = -\lambda_1 \alpha_1 e^{-\lambda_1 t} - \lambda_2 \alpha_2 e^{-\lambda_2 t}$$

and

$$m_{11} x_1^2 = m_{11} \left[\alpha_1^2 e^{-2\lambda_1 t} + 2\alpha_1 \alpha_2 e^{-(\lambda_1 + \lambda_2)t} + \alpha_2^2 e^{-2\lambda_2 t} \right]$$

and

$$\int_0^\infty m_{11} x_1^2 dt = m_{11} \left[\frac{\alpha_1^2}{2\lambda_1} + \frac{2\alpha_1 \alpha_2}{\lambda_1 + \lambda_2} + \frac{\alpha_2^2}{2\lambda_2} \right] \qquad \text{(B.2)}$$

$$m_{22} x_2^2 = m_{22} \left[\lambda_1^2 \alpha_1^2 e^{-2\lambda_1 t} + 2\lambda_1 \lambda_2 \alpha_1 \alpha_2 e^{-(\lambda_1 + \lambda_2)t} + \lambda_2^2 \alpha_2^2 e^{-2\lambda_2 t} \right]$$

then

$$\int_0^\infty m_{22} x_2^2 dt = m_{22} \left[\frac{\lambda_1^2 \alpha_1^2}{2\lambda_1} + \frac{2\lambda_1 \lambda_2 \alpha_1 \alpha_2}{\lambda_1 + \lambda_2} + \frac{\lambda_2^2 \alpha_2^2}{2\lambda_2} \right] \qquad \text{(B.3)}$$

$$m_{12} x_1 x_2 = -2m_{12} \left[\lambda_1 \alpha_1^2 e^{-2\lambda_1 t} + \lambda_2 \alpha_2^2 e^{-2\lambda_2 t} + \alpha_1 \alpha_2 (\lambda_1 + \lambda_2) e^{-(\lambda_1 + \lambda_2)t} \right]$$

then

$$\int_0^\infty m_{12} x_1 x_2 dt = -2m_{12} \left[\frac{\lambda_1 \alpha_1^2}{2\lambda_1} + \alpha_1 \alpha_2 + \frac{\lambda_2 \alpha_2^2}{2\lambda_2} \right] \qquad \text{(B.4)}$$

And

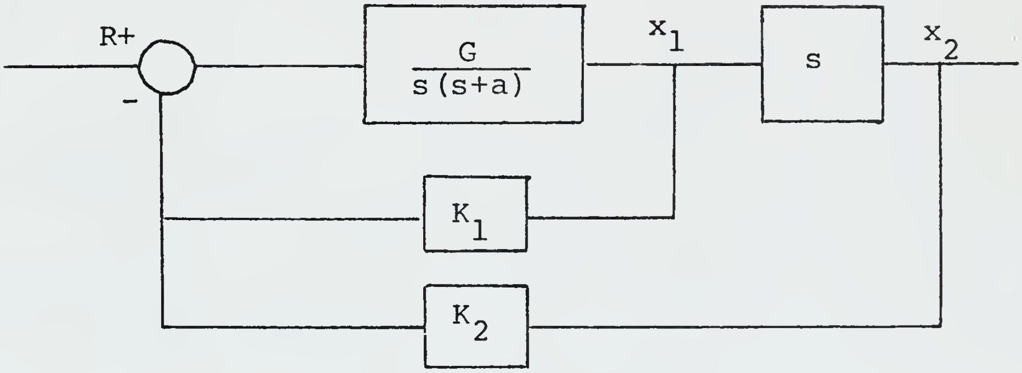
$$J = B.2 + B.3 + B.4$$

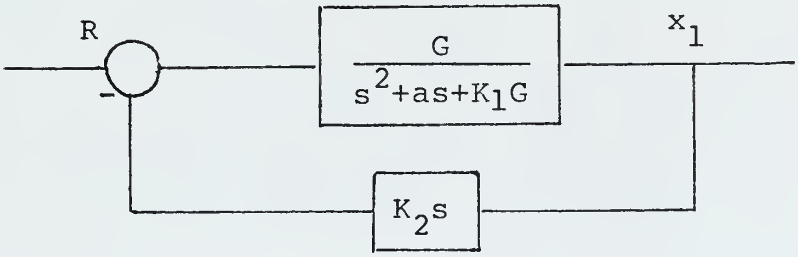
$$= m_{11}\left[\frac{\alpha_1^2}{2\lambda_1} + \frac{2\alpha_1\alpha_2}{\lambda_1+\lambda_2} + \frac{\alpha^2}{2\lambda_2}\right]$$

$$- 2m_{12}\left[\frac{\lambda_1\alpha_1^2}{2\lambda_1} + \frac{\alpha_1\alpha_2(\lambda_1+\lambda_2)}{\lambda_1+\lambda_2} + \frac{\lambda_2\alpha_2^2}{2\lambda_2}\right]$$

$$+ m_{22}\left[\frac{\lambda_1^2\alpha_1^2}{2\lambda_1} + \frac{2\lambda_1\lambda_2\alpha_1\alpha_2}{\lambda_1+\lambda_2} + \frac{\lambda_2^2\lambda_2^2}{2\lambda_2}\right] \qquad (B.5)$$

To find λ_1 and λ_2 let:





$$\frac{x_1}{R} = \frac{G}{s^2 + (a+K_2G)s + K_1G}$$

$$\omega_n = \sqrt{K_1G}$$

$$2\zeta\omega_n = a + K_2G$$

$$\therefore \quad \zeta = \frac{a + K_2G}{2\sqrt{K_1G}}$$

Then

$$s^2x_1 + 2\zeta\omega_n s x_1 + \omega_n^2 x_1 = GR$$

Let

$$x_1 = x_1 \quad , \quad x_2 = \dot{x}_1$$

$$\therefore \ \dot{x}_2 = -2\zeta\omega_n x_2 - \omega_n^2 x_1 + GR$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} R$$

when R = 0

$$\underset{\sim}{x} = \begin{bmatrix} sI-A \end{bmatrix}^{-1} \underset{\sim}{x}(0)$$

$$\begin{bmatrix} sI-A \end{bmatrix} = \begin{bmatrix} s & -1 \\ \omega_n^2 & s+2\zeta\omega_n \end{bmatrix}$$

$$\begin{bmatrix} SI-A \end{bmatrix}^{-1} = \frac{1}{s^2+2\zeta\omega_n \ s+\omega_n^2} \begin{bmatrix} s+2\zeta\omega_n & 1 \\ -\omega_n^2 & s \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{s^2+2\zeta\omega_n s+\omega_n^2} \begin{bmatrix} s+2\zeta\omega_n & 1 \\ -\omega_n^2 & s \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$x_1 = \frac{(s+2\zeta\omega_n)x_1(0)+x_2(0)}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{(s+\lambda_1+\lambda_2)x_1(0)+x_2(0)}{(s+\lambda_1)(s\rightarrow\lambda_2)}$$

$$x_2 = \frac{-\omega_n^2 x_1(0) + sx_2(0)}{s^2+2\zeta\omega_n s\rightarrow\omega_n^2}$$

$$x_1 = \frac{\alpha_1}{s+\lambda_1} + \frac{\alpha_2}{s+\lambda_2}$$

where

$$\lambda_1 = \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$$

$$\lambda_2 = \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$$

$$\alpha_1 = \left.\frac{(s+2\zeta\omega_n s)x_1(0) + x_2(0)}{s+ \quad {}_2}\right|_{s=-\lambda_1}$$

$$= \frac{(-\lambda_1+2\zeta\omega_n)x_1(0)+x_2(0)}{\lambda_2-\lambda_1}$$

$$\alpha_1 = \frac{\lambda_2 x_1(0) + x_2(0)}{\lambda_2-\lambda_1} \qquad \text{(B.6)}$$

Similarly:

$$\alpha_2 = \frac{\lambda_1 x_1(0) + x_2(0)}{-(\lambda_2-\lambda_1)} \qquad \text{(B.7)}$$

Note that:

$$(\alpha_1 + \alpha_2) = x_1(0)$$

From B.5:

$$\begin{aligned} \frac{J}{p_{11}} = {} & A\left[\frac{\alpha_1^2}{2\lambda_1} + 2\,\frac{\alpha_1\alpha_2}{\lambda_1+\lambda_2} + \frac{\alpha_2^2}{2\lambda_2}\right] \\ & - K_1K_2\left[\alpha_1^2+2\alpha_1\alpha_2+\alpha_2^2\right] \\ & + B\left[\frac{\alpha_1^2}{2\lambda_1^{-1}} + \frac{2\alpha_1\alpha_2}{\lambda_1^{-1}+ \quad {}_2^{-1}} + \frac{\alpha_2^2}{\lambda_2^{-1}}\right] \end{aligned} \qquad \text{(B.8)}$$

where

$$A = K_1^2 + \frac{q_{11}}{p_{11}}$$

$$B = K_2^2 + \frac{q_{22}}{p_{11}}$$

Substituting B.6 and B.7 into B.8 and rearranging

$$\frac{J}{p_{11}} = \left\{ \frac{A}{(\lambda_2-\lambda_1)^2}\left[\frac{\lambda_2^2}{2\lambda_1} - \frac{2\lambda_1\lambda_2}{\lambda_1+\lambda_2} + \frac{\lambda_1^2}{2\lambda_2}\right]\right.$$

$$+ \frac{B}{(\lambda_2-\lambda_1)^2}\left[\frac{\lambda_2^2}{2\lambda_1^{-1}} - \frac{2\lambda_1\lambda_2}{\lambda_1^{-1}+\lambda_2^{-1}} + \frac{\lambda_1^2}{2\lambda_2^{-1}}\right]$$

$$\left. - K_1K_2 \right\} x_1^2(0)$$

$$+ \left\{ \frac{A}{(\lambda_2-\lambda_1)^2}\left[\frac{\lambda_2}{\lambda_1} - 2 + \frac{\lambda_1}{\lambda_2}\right]\right.$$

$$\left. + \frac{B}{(\lambda_2-\lambda_1)^2}\left[\frac{\lambda_2}{\lambda_1^{-1}} - \frac{2(\lambda_1+\lambda_2)}{(\lambda_1^{-1}+\lambda_2^{-1})} + \frac{\lambda_1}{\lambda_2^{-1}}\right]\right\} x_1(0)x_2(0)$$

$$+ \left\{ \frac{A}{(\lambda_2-\lambda_1)^2}\left[\frac{1}{2\lambda_1} - \frac{2}{(\lambda_1+\lambda_2)} + \frac{1}{2\lambda_2}\right]\right.$$

$$\left. + \frac{B}{(\lambda_2-\lambda_1)^2}\left[\frac{1}{2\lambda_1^{-1}} - \frac{2}{\lambda_1^{-1}+\lambda_2^{-1}} + \frac{1}{2\lambda_2^{-1}}\right]\right\} x_2^2(0) \quad \text{(B.9)}$$

Substituting

$$\lambda_1 = \zeta\omega_n + \omega_n\sqrt{\zeta^2-1} = a + b$$

$$\lambda_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2-1} = a - b$$

$\lambda_1\lambda_2 = a^2-b^2$ into B.9.

$$\frac{J}{p_{11}} = \left\{ \frac{A}{4b^2}\left[\frac{(a-b)^2}{2(a+b)} - \frac{2(a^2-b^2)}{2a} + \frac{(a+b)^2}{2(a-b)}\right]\right.$$

$$+ \frac{B}{4b^2}\left[\frac{(a-b)^2(a+b)}{2} - \frac{2(a^2-b^2)^2}{2a} + \frac{(a+b)^2(a-b)}{2}\right]$$

$$\left. -K_1K_2 \right\} x_1^2(0)$$

$$+ \left\{ \frac{A}{4b^2} \left[\frac{a-b}{a+b} - 2 + \frac{a+b}{a-b} \right] \right\} x_1(0)x_2(0)$$

$$+ \left\{ \frac{A}{4b^2} \left[\frac{1}{2(a+b)} - \frac{2}{2a} + \frac{1}{2(a-b)} \right] \right.$$

$$\left. + \frac{B}{4b^2} \left[\frac{a+b}{2} - \frac{2(a^2-b^2)}{2a} + \frac{a-b}{2} \right] \right\} x_2^2(0) \qquad \text{(B.10)}$$

Adding and rearranging

$$\frac{J}{p_{11}} = \left\{ \frac{A}{4} \left[\frac{5a^2-b^2}{a(a^2-b^2)} \right] + \frac{B(a^2-b^2)}{4a} - K_1K_2 \right\} x_1^2(0)$$

$$+ \left\{ \frac{A}{a^2-b^2} \right\} x_1(0)x_2(0)$$

$$+ \left\{ \frac{A}{4a(a^2-b^2)} + \frac{B}{4a} \right\} x_2^2(0) \qquad \text{(B.11)}$$

Substituting

$a = \zeta\omega_n$

$b = \omega_n \sqrt{\zeta^2-1}$ into B.11

$$\frac{J}{p_{11}} = \left[A(1+4\zeta^2) + B\omega_n^2 - 4\zeta\omega_n K_1K_2 \right] \frac{x_1^2(0)}{4\zeta\omega_n}$$

$$+ \frac{A}{\omega_n^2} x_1(0)x_2(0)$$

$$+ \left[\frac{A}{\omega_n^2} + B \right] \frac{x_2^2(0)}{4\zeta\omega_n}$$

But

$$A = K_1^2 + \frac{q_{11}}{p_{11}}$$

$$B = K_2^2 + \frac{q_{11}}{p_{11}}$$

$$\omega_n^2 = K_1K$$

$$2\zeta\omega_n = p+K_2K$$

Then

$$\frac{J}{p_{11}} = \left\{ \left(K_1^2 + \frac{q_{11}}{p_{11}} \right) + \frac{(K_1^2 + \frac{q_{11}}{p_{11}})(p+K_2K)^2}{K_1K} + K_1K\left(K_2^2 + \frac{q_{22}}{p_{11}} \right) - 2K_1K_2(p+K_2K) \right\} \frac{x_1^2(0)}{2(p+K_2K)}$$

$$+ \frac{K_1^2 + \frac{q_{11}}{p_{11}}}{K_1K} \; x_1(0)x_2(0)$$

$$+ \left\{ \frac{K_1^2 + \frac{q_{11}}{p_{11}}}{K_1K} + K_2^2 + \frac{q_{22}}{p_{11}} \right\} \frac{x_2^2(0)}{2(p+K_2K)} \qquad \text{(B.12)}$$

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13. ABSTRACT

Parameter plane studies of an optimal second order regulator are presented. Emphasis is placed on the interpretation of the cost function and the sensitivity of the cost function to plant parameter incremental variations. An analysis is made of cost functions weighting factors and their effect on damping, speed of response, and cost.

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